Oscillations

by

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A rendered version of the Lorenz attractor made with Chaoscope 0.3
(www.chaoscope.org)

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1. Introduction

1.1. Historical introduction to dynamics

How systems evolve in time has always interested mankind. However, only in the 16th Century it became possible to solve complicated problems of dynamics thanks to Sir Isaac Newton. With the invention of calculus a new era of physics began. Using this new theory of mathematics, many “old” problems were solved. People thought that with this new mathematics and physics one could do almost everything, but in the same time it opened a wide field of even more complicated problems. It turned out that even as simple sounding things like the “Three-Body-Problem” were near to impossible to unravel. It took quite a long time till there was an answer found. In the 1890s Poincaré used a geometric approach that gave birth to the modern field of dynamics. The prize for this instrument was that he, too, could not quantitatively solve those problems but he was able to explain them qualitatively. Poincaré was also the first one to assume a thing like chaos in a physical way. But this should become important only at about 60 years later.

These new dynamics led to the invention of the radio, the radar and the laser, thanks to the appliance of non-linear oscillators in physics and engineering. The theory behind this topic was largely developed by people like van der Pol, Andronov, Littlewood, Cartwright, Levinson and Smale. The embedding of those theories in classical mechanics we owe to Birkhoff, Kolmogorov, Arnol’d and Moser who were also the ones to find and prove the powerful KAM-theorem.

In 1963 the first real step into the world of chaos was made by Edward Norton Lorenz who found the first Strange Attractor.

With the help of computers it was now possible to investigate and experiment further into that matter. In the 1970s several new astounding discoveries were made, like the finding of chaos in biological systems or the discovery of fractals by Mandelbrot. During the 80s interest in chaos grew more and more, and with faster computers even more could be studied. And even today we still do not come near to unravelling all the questions and mysteries concerning fractals, oscillations and chaos.
1.2 Basics of oscillations

An oscillation is an temporal periodic movement which happens when a system is pushed out of a mechanic, thermionic or electric balance and is forced to go back into its balanced situation. Simple examples of oscillations are a jumping ball, a swinging pendulum and the movement of an attracted spring.

1.2.1 Period and frequencies

The period $T$ says that a resolute situation is repeating after a definite time called $T$ which means:

$$u(t+T) = u(t)$$

The SI unit of time is seconds (s).

The frequency $f$ counts the number of oscillations happen over a unit time

$$f = \frac{1}{T}$$

This means that if the time is measured in s the unit for $f$ is Hertz (Hz).

1.2.2 Amplitude

The amplitude is the point of the system where the system reaches a maximal distance to the stable point. By an oscillating spring for example the amplitudes are places where the mass is switching the direction of movement.

1.2.3 Simple harmonic oscillator

An example we use a spring which on the one side hangs free at a fixed point and at the other side hangs a moveable mass point. The spring has no mass and the amplitude is not too large. When the spring is not oscillating the mass is at the point zero and the movement is only on one axis. When you now pull at the spring and leave it alone it begins to oscillate through its equilibrium (see figure 1.1).

figure 1.1. : The mass attached to the spring is oscillating around its stable point.
The normal equations for the returning force in the spring is Hooke’s law:

\[ F = -kx \]

Because of second Newton law we substitute the force with \( m \ddot{x} \). And so the differential equation for the movement is:

\[ m \ddot{x} = -kx \]

These equation have a partial solution with the initial condition \( x_0 \) by time \( t = 0 \):

\[ x(t) = x_0 \cdot \cos(\omega t) \]

with:

\[ \omega = \sqrt{\frac{k}{m}} \]

The \( \omega \) is the circle frequency and the \( x_0 \) is the amplitude. With the help of \( \omega \) you can calculate the period \( T \) and so also the frequency \( f \):

\[ T = \frac{2\pi}{\omega} \]

With this solution we can obtain that the frequency only depends on the spring constant \( k \) and the mass \( m \). The amplitude is completely independent to the characteristics of the motion its only depends on the initial conditions.
2. Van-der-Pol Oscillator (VdPO)

Introduction

In this Part we study different types of oscillations using the Van-der-Pol oscillations.

The electric circuits (fig 2.1) the VdP equations are based on played a important role in early radios. The non-linearity in the system led to deeper analysis of those problems. The VdP circuit is a simple LCR-circuit with a non-linear Resistor. Since this System is self-sustaining there are three important elements in the circuit. We’ve got the LCR circuit which produces the damped oscillation. The inductive coil is helping to keep the oscillation alive by inducting energy into the system through feedback. The Last part is the triode (a three-diode-lamp). The Triode is damping for High currents and is working as a source for Low Currents and thus bringing the oscillation to a stable state. We have used a computer simulation (dynamics solver) to analyse different forms of oscillations the VdP equation can be used to describe.

\[ \ddot{x} - a\dot{x}(1 - x^2) + x = 0 \]  

eq 2.1

fig 2.1-- circuit diagram of a VdP circuit
2.1 Simple Harmonic Oscillator

In the case that the nonlinear parameter $a = 0$, the system is reduced to a simple harmonic oscillator.

$$\ddot{x} + x = 0$$  \hspace{1cm} \text{eq 2.2}

This equation can be solved analytically and its solution is periodic. The amplitude and the initial phase are determined by the initial conditions we chose. However, in the $x\dot{x}$ – phasespace we only obtain circles. In Fig.2.2. we show the phase portrait with different initial conditions.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.5</th>
<th>-0.8</th>
<th>-0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$</td>
<td>0.25</td>
<td>0</td>
<td>0.13</td>
</tr>
<tr>
<td>colour</td>
<td>Red</td>
<td>Blue</td>
<td>Green</td>
</tr>
</tbody>
</table>

In order to see the phase-shift we plot the time series (Fig.2.3.) It is clear to see the phase-shift between three different realizations.
2.2 Self-sustained system

By changing the parameter $a$ to $a = 1$ we can slowly introduce the non-linear damping term. The orbit is not a circle anymore. For various initial conditions it is always attracted to the same orbit. This orbit is called the limit cycle. It is called isolated because it has a closed orbit, while all the other orbits are attracted by this limit cycle. You can see that trajectories with Initial Conditions inside the limit cycle are attracted to the limit cycle as well as trajectories with initial conditions outside of the cycle. This point is called stable (fig 2.4).

![Phasespace](image1.png)

**fig 2.4: $\dot{x}x$ – phasespace for the self-sustaining oscillator**

The time series shows the phase-shift as well as the amplitude all the three realizations have.

![Time Series](image2.png)

**fig 2.5 time series for the self-sustaining oscillator**
Three important properties of self-sustained systems are:

1) It contains an internal power source which provides energy to the system. This energy is transformed into oscillatory movement, until the energy source is depleted. Such systems are called autonomous which means the oscillator is not explicit time-dependant.

2) Only the parameters determine the form of oscillation. The system does not depend on the transient.

3) The oscillation is stable for small perturbations, so the oscillation returns to its parameter determined shape.

Our internal power source in the VdP circuit is the inducting coil I mentioned before. The work of this power source can be seen in the green oscillation, which reaches the limit cycle after its first period. The damping can be seen in the blue and the red example. Here the oscillator dissipates energy until it reaches the limit cycle.

We can also see in fig 2.4 that all oscillations are parameter determined because all oscillations are attracted by the limit cycle and, after the transient.

The figure 2.6 is showing the effect if we increase the parameter \(a\) even more. The orbits become distorted from the simple harmonic circle.
2.3 Relaxation Oscillation

For an even larger parameter $a$ ($a = 10$ in this case) you can see an oscillation called the relaxation oscillation. It is characterized by a slow build-up followed by a fast discharge. You can see those two parts of the oscillation in the phase space. In the time series (fig 2.8) you can see it even better and in addition the different times needed for charging and discharging.

![Phase space for the relaxation oscillation](image1)

![Time series for relaxation oscillation](image2)
2.4 Hard Excitation

Now we will briefly take a look at another approximation for the anode current of the VdPO. We analysed the problem for an approximation with a polynomial of an order up to 5. In this case the VdP equation reduces to

\[ \ddot{x} - \alpha (1 - \beta x^2 - \theta x^4) \dot{x} + x = 0 \]

With the parameters \( \theta = -27, \alpha = -0.1, \beta = 20 \) we got a system with a different behaviour. Different initial Conditions in figure 2.9 showed us three fixed states.

This is a complex system and hard to solve analytically but numerical experiment lets us analyse the system qualitatively.

If the initial conditions are inside the unstable limit cycle the trajectory (green) is attracted by the stable fixed points in the centre of the phase space. For initial conditions outside the unstable limit cycle the trajectory (red) is attracted by the outer stable limit cycle. This limit cycle also attracts trajectories (blue) with initial conditions outside of the stable limit cycle.

From the above analysis we see that there should, in principle, exist a limit circle between the green and red colour, which is unstable, dividing the phase space into different regions.

fig 2.9 phase space for hard excitation
initial conditions \((\dot{x}, x):\) red-(0;0.49), green-(0;0.48), blue-(1.2;1)
3. From Theory to Experiments

In the previous section we talked about oscillations theoretically. In this section we discuss electronic experiments where oscillations occur. In these experiments we use especially capacitors (C) and resistances (R) that produce oscillations.

3.1 Basics

To understand the following experiments it is important to know what capacitors are and how do they work. Capacitors are passive electronic components that store energy in the form of an electrostatic field. The electrostatic field arises between conductive plates while charging. Often there is an insulating material called dielectric between them that changes characteristics of capacitors. By charging the voltage increases. The ratio of the charge magnitude on each plate to the electric potential (voltage) between the plates is known as the capacitance of the capacitor (C). The capacitance of a device depends mostly on the plate geometry and the nature of the dielectric.

The capacitance of a capacitor can be calculated if the geometry of the conducting electrode and the dielectric properties of the insulator between the conductors are known. For example, the capacitance of a parallel-plate capacitor constructed of two parallel plane electrodes of area $A$ separated by a distance $d$ is approximately equal to the following:

$$C = \varepsilon \frac{A}{d}$$

- $C$ is the capacitance in farads, \(1\ F = 1\ C/V\)
- $\varepsilon$ is the permittivity of the insulator used (or $\varepsilon_0$ for a vacuum)
- $A$ is the area of each plane electrode, measured in square metres
- $d$ is the separation between the electrodes, measured in metres

**Current and Voltage Characteristics of Capacitors**

By applying voltage (switch position 1 in fig.4) to an electric circuit that includes a capacitor the potential on the capacitor increases slowly while the current is decreasing slowly. The voltage on the capacitor increases at first rapidly and then slowly until the capacitor is charged. After switching off the voltage (switch position 2 in fig.4) the capacitor discharges and the voltage decreases and the current increases. This is shown in

---

fig 1: Capacitor

fig 2: $I(t)$ in capacitors
The plot in fig 2 shows a jump between charging and discharging. This is deduced from the direction the current flows. At first it flows in one direction while charging. By switching off the voltage the electromotive force is the capacitor. The electrons flow in the reverse direction. Hence the plot goes to the minus part.

Charging and discharging can be described as two simple homogeneous differential equations. Corresponding to Kirchhoff's and Ohm's law we get the equation

\[
\frac{dI}{dt} = \frac{1}{RC} \cdot I
\]

for charging. The solution for this equation is

\[
I(t) = I_0 \exp\left(-\frac{t}{RC}\right)
\]

To get the equation for the voltage we use Kirchhoff's law:

\[
U_c(t) = U_0 - U_r
\]

\[
= U_0 - R \cdot I_0 \exp\left(-\frac{t}{RC}\right)
\]

\[
= U_0 \left(1 - \exp\left(-\frac{t}{RC}\right)\right)
\]

The equation for discharging (current) is:

\[
I(t) = I_0 \left(1 - \exp\left(-\frac{t}{RC}\right)\right)
\]

To get the differential equation for discharging we need to assume that the current

\[
I(t) = \frac{U(t)}{R}
\]

is flowing through the capacitor. The current is the rate of change of the charge in the capacitor. Therefore we get the differential equation
\[ \frac{dU_c}{dt} = -\frac{U_c}{RC} \]

So the solution for discharging is:

\[ U_c(t) = U_0 \exp\left( -\frac{t}{RC} \right) \]

A good example where we use a capacitor in DC circuits is a flash light of a camera. The battery loads capacitor until it's fully charged and then the capacitor discharges through the lamp.

**Capacitors in AC circuits**

So far we studied how capacitors work with DC. In DC circuits capacitors have an infinite resistance. The capacitor charges until its capacitance is reached. Then no electrons are moving any more. However in AC-circuits capacitors have resistance that depends on the frequency of the input signal. The greater the frequency the lower the resistance. The resistance is called capacitive reactance.

We used the circuit in fig.5 to get the dependence of resistance and frequency. The voltage over a capacitor is given by

\[ U = \frac{Q}{C} \]

and derivating by time we get

\[ \frac{dU}{dt} = \frac{d}{dt} \left( \frac{Q}{C} \right) = \frac{I}{C} \]

\( U_c \) is alternating so

\[ U_c(t) = U_0 \cos(\omega t) \]

and

\[ \frac{dU_c}{dt} = -U_0 \sin(\omega t) \]

\[ \rightarrow I(t) = -U_0 C \omega \sin(\omega t) \]

\( \omega \) – frequency

\[ \rightarrow I(t) = U_0 C \omega \cos(\omega t + \pi) \]

\[ \rightarrow I(t) = I_0 \cos(\omega t + \pi) \]

Comparing \( U(t) \) and \( I(t) \) we recognize that the current leads the voltage by \( \pi \). This is one important property of capacitors in alternating currents.

Now it is easy to find the equation for the reactance.

Using Ohm's law \[ |Z| = \frac{U_0}{I_0} \] and the relation \( I_0 = U_0 C \omega \) we figure out that

\[ Z = x_c = \frac{1}{\omega C} \]

This is the impedance only for our circuit.

The dependency of the frequency and the reactance is often used for example to filter specific frequencies that we examine in another part.
3.2 Relaxation Oscillation

The relaxation oscillation is a type of self-sustained oscillations. In this section we study these oscillation through an experiment. The circuit is shown in the following figure.

![Figure 6: Circuit diagram](image)

It is comprising of DC power supply, a resistance a Capacitor C and a lamp(G) with internal resistance($R_v$) to reduce current through the lamp. The glow lamp consists an anode and cathode filled with gas.

At first we were looking for the ignite voltage of the glow lamp. For this part we used only the glow lamp and the resistor to regulate the voltage. The gas begins to glow when the input voltage has the same amount as the ignition voltage of the glow lamp. The ions of the gas are discharging by emitting light.

The extinction voltage is lower than the ignition voltage. That is because the ions are heated so they have more energy. At the point of ignition they are not heated so the electrons need more energy to make them glowing.

The ignition voltage in our examination was $U_i = 145.6\, \text{V}$ and the extinction voltage was $U_{ex} = 133.2\, \text{V}$.

So far the behaviour of capacitor and glow lamp is discussed we explain the functionality of the circuit in fig 6.

At the beginning the capacitor charges over the resistance and the lamp does not glow. The voltage of the glow lamp increases while the capacitor is charging. When the capacitor reaches the ignition voltage the lamp ignites. The internal resistor($R_v$) of the lamp falls and the capacitor discharges. In addition to this the voltage over the lamp goes down until the extinction voltage and the lamp goes out but then the capacitor charges again. This goes on until stopping the process. This process is a stable oscillation and also self-sustained. In this experiment we measured the period for different capacitors and resistances. So we want to examine what happens with the oscillation.

There are two different processes in the entire process of oscillation. The charging and discharging process. While looking at the charging process we can find the voltage over the lamp is

$$U_{gl} = U_i + (U_i - U_e) \exp\left(-\frac{t}{RC}\right)$$
and the charging time is

\[ t_c = R_C \cdot \ln \left( \frac{U_e - U_i}{U_e - U_l} \right) \]

This works because we assume that \( R_v \) is infinite and so the capacitor charges as usual until it reaches the ignition voltage.

The second process is the flashing and discharging. The time until the lamp flashes is

\[ t_f = R_v \cdot C \cdot \ln \left( \frac{U_i - U_0}{U_i - U_0} \right) \]

So we can find the period \( T \) by adding \( t_i \) and \( t_c \).

The lamp only glows if the overall voltage is higher than or equal to the ignition voltage.

\( U_0 \) is in our examination 131.02 V. We got this value by plotting the \( U(I) \) curve and applying a regression curve. The equation for the linear regression is \( U(I) = 19.07 \text{ V/mA} \cdot I + 131.02 \text{ V} \). The first value is \( R_v \) and the second our \( U_0 \).

With \( U_0, U_i, U_l, R_v, U_e = 160 \text{V} \) and given \( C \) we can calculate the period.

The table shows some calculated and measured values.

<table>
<thead>
<tr>
<th>( R/\Omega )</th>
<th>( C/\mu\text{F} )</th>
<th>measured ( T/\text{s} )</th>
<th>calculated ( T/\text{s} )</th>
<th>Chuntering/%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>0.44</td>
<td>0.34</td>
<td>28.79</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.58</td>
<td>0.65</td>
<td>13.61</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.13</td>
<td>1.31</td>
<td>15.99</td>
</tr>
<tr>
<td>2.5</td>
<td>2</td>
<td>2.61</td>
<td>3.17</td>
<td>21.54</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5.42</td>
<td>6.28</td>
<td>15.85</td>
</tr>
</tbody>
</table>

As you can see by increasing the capacity and the resistance, the time period \( T \) of the oscillation increases as well.
3.3 High- and low pass filter
In the following experiment we use the properties of RC circuits when connected to alternating current. We examine the high pass and the Low Pass filter that is used to filter high or low frequencies. The pictures show the circuit of both.

A high-pass filter is a filter that passes high frequencies well, but attenuates or reduces frequencies lower than the cut-off frequency. The actual amount of attenuation for each frequency varies from filter to filter. The opposite of a high-pass filter is a low-pass filter that passes low frequencies well but inhibits or reduces the amplitude of frequencies greater than the cut-off frequency.

In the figures \( U_e \) is the input voltage and \( U_a \) is the output voltage.

**Low-pass filter**
The filter can be seen as a voltage divider that depends on the frequency. Using the equation for a voltage divider we get for the output voltage

\[
|U_a| = \frac{|U_e|}{R + \frac{|X_c|}{|X_c|}} = \frac{U_e}{\sqrt{1 + \omega^2 R^2 C^2}}
\]

In the pictures we recognize a phase shift that can also be calculated with the equation

\[
\Phi = \arctan(-\omega RC)
\]

In the diagram you can see what a low-pass filter does with high frequency. The input voltage \( U_e = 2.0 \text{V} \) and the output voltage is only \( U_a = 0.1 \text{ V} \) with a phase shift of approx. 90°.

**fig 8: high pass circuit**
**fig 9: low pass circuit**

**fig 10: Input- & output signal when a high frequency is given to a low-pass filter**
In the next diagram you can see lower frequency on a low-pass filter.

![Low-pass filter - low frequency](image)

**fig 11: Input- & output signal when a low frequency is given to a low-pass filter**

This signal is much attenuated as well but not as much as the attempt with higher frequency. The input voltage is the same but the output voltage has changed to $U_a=0.15\, \text{V}$. The frequency isn't low enough therefore it is attenuated so much. Using lower frequencies as we did would give an output signal with greater amplitude.

**High-pass filter**

To get the output voltage of a high-pass filter we can go the same way.

$$|U_a| = \frac{R|U_e|}{R+|X_c|} = \frac{|U_e|\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$$

and the phase shift is

$$\Phi = \arctan \left( \frac{1}{\omega RC} \right)$$

. Here you can see what a high-pass filter does with relatively low frequency. You can see that the amplitude of $U_a$ is attenuated but not much.
The input voltage is $U_e=2.5\,\text{V}$ and the output voltage $U_a=1.5\,\text{V}$.

Then we examined how it works with high frequencies. As you can see the output voltage was not so much attenuated. This is what we expected.

The input voltage is $U_e=2.5\,\text{V}$ and the output voltage $U_a=2.0\,\text{V}$. The phase shift is higher when the frequency is lower.

---

fig 12: Input- & output signal when a low frequency is given to a high-pass filter

fig 13: Input- & output signal when a high frequency is given to a high-pass filter

As you can see the output voltage was not so much attenuated. This is what we expected. The input voltage is $U_e=2.5\,\text{V}$ and the output voltage $U_a=2.0\,\text{V}$. The phase shift is higher when the frequency is lower.
Low – and high-pass filters are used for example in hifi-systems with many speakers. So that each speaker gets its special frequencies.
4. Synchronization

4.1 Introduction

Synchronization is a phenomena encountered as often in all fields of science as well as in daily life. But what is synchronization? Those phenomena that have to do with synchronization are very different but usually follow the same universal laws. Some examples for synchronization would be biological clocks that lead to the birds migration in fall or the oscillation of the millennium bridge in London where the pedestrian, without knowing, started to synchronize their steps until they were walking in lockstep so that the bridge had to be closed. Usually oscillating objects are synchronized because they are open systems and not isolated from their environment. A prominent example is the first investigation of the problem by Christiaan Huygens in 1665. He observed that his highly precise clocks where often oscillating with the same phase. In Huygens time the major interest was on conservative oscillators. Those oscillators cannot synchronize. A kind of irreversibility (dissipation of energy) is needed as we have in our VdP oscillator.

Oscillatory systems of that kind have a tendency to synchronize their phase. To synchronize the systems a coupling between them is needed, which is characterized by its strength and the frequency detuning (or frequency mismatch) they have. The strength of interaction isn't measurable very well in most cases. The frequency detuning $\Delta f$ says how big the difference between the frequencies $f_1, f_2$ of two uncoupled oscillatory systems is. If we measure the difference in frequency $\Delta F$ for the coupled oscillators we can get a dependency that gives us the graph fig 4.1.

![fig 4.1 Frequency vs. detuning plot for a fixed coupling strength between two oscillatory systems](image)

You can see that there is a range of frequency detuning $\Delta f$ where the difference of the coupled oscillators is $\Delta F = 0$. This range is called synchronisation region because that is where synchronisation happens.

Now synchronized systems can oscillate in phase or anti-phase. In our experiment we are only interested in 1:1 synchronization.
4.2 Synchronisation of the VdPO by external force

Now a periodic external force $A\sin(\Omega t)$ is driving the VdPO. We consider this a coupling of two oscillatory systems.

$$\ddot{x} - ax(1 - x^2) + x = A\sin(\Omega t) \quad \text{eq 4.1}$$

We numerically analysed the equation 4.1 to find the synchronization region of the coupled systems. We plotted the outputted force $A\sin(\Phi t)$ with respect to the inputted generalized position for different amplitudes $A$ which gave us lissajou figures like the figures 4.2 to 4.5.

![Lissajou figures](image)

- **fig 4.2 (left)** simple lissajou figure for frequency in synchronization region $A=1, a=1, \Omega = 1$
- **fig 4.3 (right)** lissajou figure for frequency out of the synchronization region $A=0.8, \Omega = 0.701$
- **fig 4.4 (left)** simple lissajou figure for frequency $\Omega_{\text{min}}$ $A=0.8, \Omega = 0.702$
- **fig 4.5 (left)** simple lissajou figure for frequency $\Omega_{\text{max}}$ $A=0.8, \Omega = 1.1465$
If the system is synchronized, one finds a simple lissajous figure (fig.4.2). However, if not, the lissajous figure becomes more complex. A good example is the fig 4.4 where still a synchronization occurs. If we change the frequency $\Omega$ just a little bit the figure changes to fig. 4.3. We want to find the synchronization region so we tried to find the maximal and the minimal frequency $\Omega$ of the external force for each given amplitude value $A$.

We plotted the frequencies for 5 amplitude values $A$ in fig 4.6. A linear interpolation gave us the boundaries of the grey area. Form this figure the VdPO is synchronized by the external force if the parameters are in the grey region.

![fig 4.6. synchronization region for different amplitudes](image-url)
5. Chaos

5.1. Introduction to chaos

Chaotic systems are systems, which got a sensitive dependence on their initial conditions. This means, that if you run a chaotic system with defined initial conditions you will get a certain possible set of trajectories. Now, if you change your initial conditions only slightly, there will be a totally different set of trajectories.

An example for the problems of chaos is the so called “Butterfly Effect“ in the climatology or the n-body problem in stellar systems. The most prominent example (which in fact gave this effect its name) of the Butterfly Effect is that when a butterfly flaps its wings for example in South America, it could be possible, that this influences the weather in Asia and might start a hurricane. That does not mean that every single butterfly-flap causes a storm, it just means there could be the possibility.

The n-body problem means that in a stellar system all gravitating objects influence the whole system. For example in our solar system the sun and the bigger gas planets obviously have the greatest influences on the movements of all the other objects but also the smaller moons, asteroids etc. can disturb the movement. The examination of chaotic processes in a far more detailed way became possible only in the middle of the 20th century with the invention of the computer. But even today even simple looking problems cannot be solved entirely.
5.2. The double pendulum

The pendulum is a system of two connected pendulous (s. figure 5.1).

\[ x_1 = l_1 \sin(\theta_1) \]
\[ y_1 = -l_1 \cos(\theta_1) \]
\[ x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \]
\[ x_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \]
With these coordinates you can calculate the velocities:

\[
\begin{align*}
\dot{x}_1 &= l_1 \dot{\theta}_1 \cos \theta_1 \\
\dot{y}_1 &= l_1 \dot{\theta}_1 \sin \theta_1 \\
\dot{x}_2 &= l_1 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_2 \cos \theta_2 \\
\dot{y}_2 &= l_1 \dot{\theta}_1 \sin \theta_1 + l_2 \dot{\theta}_2 \sin \theta_2
\end{align*}
\]

With these thems we can calculate the kinetic energy \( T \) and the potential energy \( V \) of the system:

\[
T = \frac{m_1}{2} l_1^2 \dot{\theta}_1 + \frac{m_2}{2} [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)]
\]

\[
V = -g [(m_1 + m_2) l_1 \cos \theta_1 + m_2 l_2 \cos \theta_2]
\]

With these two energies we get the Lagrange equation:

\[
L = \frac{m_1}{2} l_1^2 \ddot{\theta}_1 + \frac{m_2}{2} [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)]
\]

\[
+ g [(m_1 + m_2) l_1 \cos \theta_1 + m_2 l_2 \cos \theta_2]
\]

When you solve this equation you get the two differential equations:

\[
0 = g m_2 l_2 \sin \theta_2 + m_2 l_2 \ddot{\theta}_2 + m_1 l_2 \ddot{\theta}_1 \cos (\theta_1 - \theta_2) - m_2 l_1 \ddot{\theta}_2 \sin (\theta_1 - \theta_2)
\]

\[
0 = g (m_1 + m_2) l_1 \sin \theta_1 + (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos (\theta_1 - \theta_2) + m_2 l_1 l_2 \ddot{\theta}_2 \sin (\theta_1 - \theta_2)
\]

With these equations you can calculate a numerical solution for the pendulum. We used the program dynamic solver for the numerical solution of the double pendulum.
5.2.1. Solution for small elongations

For small elongation the movement of the pendulum is almost periodic which we also observed in the computer. For the simulation in the computer we take as parameters for the masses and the length $m_1 = m_2 = l_1 = l_2 = 1$ and for the gravitation $g = 9.81$. Also the initial velocities are fixed by zero. These parameters last over the hole simulation. The following pictures show the regularity we have seen by an elongation of $\theta_1 = 0.5$ and $\theta_2 = 0$:

![Figure 5.2: dynamical illustration with small elongations](image)

![Figure 5.3: time series of the variable $\theta_1(x_1)$](image)

![Figure 5.4: time series of the variable $\theta_2(x_2)$](image)
5.2.2. Solution for larger elongations

By larger elongation the double pendulum moves chaotic. For these cases we choose an elongation over a critical value we observed the following cases by an initial elongation of $\theta_1 = 1.4$ and $\theta_2 = 0.5$:
But for some large elongations it is also possible to observe stable periodic motions. We show the following figures for a particular pair of the initial values $\theta_1 = 2$ and $\theta_2 = 1.49$.

Figure 5.8: time series of the variable $\theta_1 (x_1)$

Figure 5.9: time series of the variable $\theta_2 (x_2)$

Figure 5.10: phase portrait of pendulum 1

Figure 5.11: phase portrait of pendulum 2

But for some large elongations it is also possible to observe stable periodic motions. We show the following figures for a particular pair of the initial values $\theta_1 = 2$ and $\theta_2 = 1.49$.

Figure 5.12: dynamical illustration for stable solution
5.2.3 Conclusion

The double pendulum is a conservative system, which means the energy is a constant. If the energy is low enough, it would appear to be periodic motion. If the energy is high enough, the motion is chaotic. However, in order to obtain the chaotic dynamics, we have to choose the initial conditions carefully because not all initial values would produce chaos. For some values, a periodic motion would occur as we have seen in the figure 5.12. The chaos we obtained from the dynamics solver is called Hamiltonian chaos because of the existence of a constant Hamiltonian. In Hamiltonian systems, the phase space is often a mixture of regular and chaotic regions and with the third example we found such a regular region where we did not expected such thing. This part is somehow out of the scope of our experiment.
5.3. The Lorenz system

One of the pioneers in the field of chaos theory is Edward Norton Lorenz. He coined terms like “Butterfly-Effect” and “Strange Attractor” and made vast achievements on the ground of deterministic chaos.

The expression “Butterfly-Effect” stands for the sensitivity of chaotic systems on slightly different initial conditions, based on the anecdote that a wing stroke of a butterfly in South America could cause a hurricane in Japan.

In 1963 Lorenz derived three seemingly simple equations to describe a simplified model of convection rolls in the atmosphere. These equations are now known as the Lorenz system.

The Lorenz system is given by

\[
\begin{align*}
\dot{x} &= s(y - x) \quad (1) \\
\dot{y} &= rx - y - xz \quad (2) \\
\dot{z} &= xy - bz \quad (3)
\end{align*}
\]

where s, r and b are parameters, especially s is called the Prandt number, r the Rayleigh number and b has no explicit name but has to do with the aspect ratio of the rolls.

The Rayleigh number measures how hard a system is driven in comparison to the dissipation.

The nonlinearities of the system consist of the terms \(-xz\) in the second equation and \(xy\) in the third.

For our experiments the parameters were set as follows: \(s = 10\), \(b = \frac{8}{3}\) while \(r\) was varied.

At first we want to find the fixed points of the system i.e. where \(\dot{x} = \dot{y} = \dot{z} = 0\). As we can easily see there is a fixed point for all values of s, r and b at \(x = y = z = 0\).

However there are two other fixed points:

I \(0 = s(y - x)\)

II \(0 = rx - y - xz\)

III \(0 = xy - bz\),

From I we can see that we need \(x = y\).

This yields if substituted in III \(z = \frac{x^2}{b}\). This gives II as \((r - 1)x = \frac{x^3}{b}\).

The case \(x = 0\) we have already found so we get \(b(r - 1) = x^2\).

So the two other fixed points are given by

\[x_{2/3} = y_{2/3} = \pm \sqrt{b(r - 1)}\]

\[z_{2/3} = r - 1.\]

The two fixed points \(\neq 0\) represent the left- or right-turning convection rolls. Now we need to determine the stability of those points.
To simplify things we first investigate the local stability of the origin, i.e. \( x = y = z = 0 \). Now for \( r < 1 \) all three directions head toward the origin and so the first fixed point has to be stable, as can be seen in figs. 5.2.01 – 5.2.03.

\[
\begin{pmatrix}
-s & s & 0 \\
1 & -1 & \pm \sqrt{b(r-1)} \\
\pm \sqrt{b(r-1)} & \pm \sqrt{b(r-1)} & -b
\end{pmatrix}
\]

Now we need to find the eigenvalues.

\[
\det(J_{2/3} - \lambda \mathbf{I}) = (-s - \lambda)[(-1 - \lambda)(-b - \lambda) + b(r-1)] - s[-b - \lambda + b(r-1)] = 0
\]

This yields the characteristic equation

\[
\dot{\lambda}^3 + (s + b + 1)\dot{\lambda}^2 + b(s + r)\dot{\lambda} + 2sb(r-1) = 0.
\]
For \( r = r_H = \frac{s(s + b + 3)}{s - b - 1} \) (in our case \( r_H \approx 24.74 \)) there are three eigenvalues:

\[
\lambda_1 = -1 \quad \lambda_2 = -\frac{b + s}{2} \\
\lambda_{2/3} = \pm \frac{\sqrt{2} \sqrt{bs(1 + s)}}{\sqrt{1 + b - s}}
\]

(These solutions were found with the help of Mathematica.).

As one can see for \( 1 + b - s < 0 \) the eigenvalues \( \lambda_{2/3} \) become purely imaginary.

This hints at a Hopf-Bifurcation.

When \( r \rightarrow 1 \) (for \( r < 1 \)) the origin becomes unstable and the two other fixed points become real and are stable. Such a transition is called a Supercritical Pitchfork Bifurcation.

For \( 1 < r < r_H \) the trajectories of the system head toward one or the other of the remaining fixed points, depending on the initial conditions.

\[\text{fig. 5.3.04: (above)}\]
For \( 1 < r < r_H \), there is also a transient and a damped oscillation, but much weaker damped now, before the trajectories come to rest at one of the nonzero fixed points.

\[\text{fig. 5.3.05: (left)}\]
In the x-y-z-diagram one can clearly see the oscillation around one of the two nonzero fixed points.
Now the question is what happens for $r > r_{\mu}$. All trajectories remain in a bounded area without any stable points or limit cycles. This is what Lorenz found out. So there had to be some “strange” thing that confined the trajectories without producing periodic motion. Lorenz termed it a “strange attractor”.

The x-z-projection of the Lorenz attractor gives only hints as to the shape of the three-dimensional form of that strange attractor, but one can already recognize the complexity of that “thing”.

Here one can clearly see the jumps from the left to the right lobe and vice versa and the different number of oscillations around each of the fixed points. One cannot foresee when the trajectory will jump back, hence the term chaotic motion.

**fig. 5.3.06:** $x_0 = 0.1$, $y_0 = z_0 = 0$

**fig. 5.3.07:** $x_0 = 0.1$, $y_0 = z_0 = 0$

**fig. 5.3.08:** In three dimensions the complex structure of the attractor reveals itself.
Nowadays strange attractors and other fractals are no longer bound to laboratories and physics alone. Art has developed a completely new branch, where fractals appear in every form and colour. Amazing pictures are to be found (see for example front cover).

One of the properties of chaos is the sensitivity on slightly different conditions. Trajectories starting at almost the same point will take completely different routes around the attractor. The difference will be observable in finite time. The images figs. 5.2.13 – 5.2.18 on the next page show the x-t-diagrams for three slightly different initial conditions and two time-intervals.

One can clearly see the differences of the motion of a particle starting every time with nearly the same initial conditions. At first it seems to take the same course, but after some oscillations, approximately at a point $t_{critical}$, the trajectories diverge. In the corresponding diagrams of the second time-interval the differences become clearer. (The time $t$ is measured in time-units used by dynamic-solver for the simulation.)
Now, how do these results correspond to the different states of motion of a heated fluid?
For $r < 1$ the heat is conducted upward without motion of the fluid. For $1 < r < 24.74$ appear convection rolls which correspond to spirals rotating clockwise around one of the nonzero fixed points and counter-clockwise around the other. For still higher $r$ the fluid motion becomes chaotic as seen in the Lorenz system.
6. Conclusion

Oscillations are part of our world. We encounter them everyday when we watch TV, listen to the radio or use the computer and so on. In biology models of oscillations help us to understand complex connections in nature (e.g. the prey-predator-model). Physics has in some parts become the science of solving oscillator-equations. Also in mathematics the chaos theory provides a huge field of research.

A future outlook for the use of oscillations may be new encryption technologies using synchronized chaos, new mediums for saving data (using holographic devices) and a lot more.
7. References


[10] Instruction manuals for the experiments