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Abstract

Three major differences to the benchmark model of D’Aspremont, Gabszewicz and Thisse (1979) characterize Puu (2002). Two of them render the setting more realistic, while the third – a different concept of stability – is not useful. Stability is only found due to a flaw in the calculations. Nonetheless, the search for a better concept of stability is imperative because the conventional concept exhibits an inconsistent informational structure.

1 Introduction

Hotelling (1929) put forth the famous ‘principle of minimum differentiation’. According to this principle two firms will locate close together in the center of a bounded linear market area with evenly distributed inelastic demand. This principle was considered as universal, and has been transferred to many applications such as the maximization of votes by political parties. It was not until 50 years later that D’Aspremont et al. (1979) proved that there was an error in Hotelling’s formal analysis. As firms move towards the center, they eventually reach a point where they can increase their profits by undercutting the rival, the equilibrium becomes unstable. D’Aspremont et al. (1979) then proposed a transportation cost function that is quadratic in distance in order to re-

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establish stability of the location game. Their solution was characterized by ‘maximum differentiation’ of the duopolists’ locations, however\(^1\).

Puu (2002) aims at deriving a stable solution to the location game employing the original assumption of linear transportation costs, but with elastic demand functions. Altogether, the changes with respect to D’Aspremont et al. (1979) increase realism to a sizeable extent. Another important difference to related literature is that Puu (2002) employs a more ambitious concept of stability. Firms’ locations are between the quartiles and the center of the market, which confirms Hotelling’s intuition (Puu, 2002, p. 2).

Unfortunately, Puu’s solution, as it stands, is wrong. The first purpose of this paper is to show why we come to this conclusion. The second purpose is that we want to demonstrate that the case considered by Puu is nonetheless interesting because – employing the conventional concept of stability – the derived equilibrium is stable, and relies on less unrealistic assumptions than related approaches. Thirdly, we want to point at an inconsistency in the usual condition for a stable equilibrium.

2 Basic set-up and duopoly profits

Let us first examine why the duopoly equilibrium in Puu (2002) is not stable, using the same terminology. Assume that demand for a homogenous commodity is distributed evenly on the interval \([-1, 1]\) at unit density. Two identical firms, \(i = 1, 2\) choose the pair of mill price and location \((p_i, x_i)\) that maximizes their respective profits \(\Pi_i\). Without loss of generality we assign \(x_1 \leq x_2\). Marginal costs of production are \(c\), and there are no fixed costs. All consumers have the same linear demand function

\[
f(z) = \begin{cases} 
\alpha - \beta z & \text{if } z < \frac{\alpha}{\beta} \\
0 & \text{if } z \geq \frac{\alpha}{\beta}
\end{cases}
\]

(1)

where \(z\) denotes the delivered price (mill price plus transportation costs) at the location of the respective consumer. Transportation costs per unit quantity of the good and per unit distance are constant and denoted \(k\). At the inner market boundary \(\hat{x}\) delivered

\(^1\)Of course, we are aware of the many refinements of this result. For instance, Economides (1986) showed that there is a range of convex transportation cost functions leading to stable equilibria between the quartiles and the ends of the market. Friedman and Thisse (1993) reestablished minimum differentiation as a result when firms collude partially. We chose to use D’Aspremont et al. (1979) as a reference since their model is most influential on subsequent literature in regional science and industrial economics.
prices of both firms are equal, \( p_1 + k(\hat{x} - x_1) = p_2 + k(x_2 - \hat{x}) \), which yields
\[
\hat{x} = \frac{p_2 - p_1}{2k} + \frac{x_1 + x_2}{2}
\] (2)

If we exclude the case of two (adjacent) monopolies, demand at the inner market boundary, as well as at both ends of the market must be greater than zero, i.e. delivered prices must be below the prohibitive price \( \alpha/\beta \) throughout the entire market.

Profits of firm 1 read
\[
\Pi_1 = (p_1 - c) \int_{-1}^{\hat{x}} (\alpha - \beta p_1 - \beta |x - x_1|) dx
\] (3)

Substituting \( \hat{x} \) by eq. (2), differentiating and solving with respect to \( x_i \) and \( p_i \), and performing an analogous procedure for firm 2 yields the following simultaneous system of equations\(^2\):
\[
x_1 = \frac{x_2 - 4}{5} + \frac{p_2 - 3p_1}{5k} + \frac{2\alpha}{5\beta k}
\] (4)
\[
x_2 = \frac{x_1 + 4}{5} + \frac{3p_2 - p_1}{5k} - \frac{2\alpha}{5\beta k}
\] (5)
\[
p_1 = \frac{1}{9} \left( 3c + 2kx_2 + 2p_2 + 4 \frac{\alpha}{\beta} + 2k \right) - \frac{\sqrt{2\lambda_1}}{18}
\] (6)

with
\[
\lambda_1 = 17(p_2 + k + kx_2)^2 - 24c \frac{\alpha}{\beta} + 18c^2 + 23 \left( \frac{\alpha}{\beta} \right)^2 - 12c(p_2 + k + kx_2) - 22 (p_2 + k + kx_2) \frac{\alpha}{\beta}
\]
\[
p_2 = \frac{1}{9} \left( 3c - 2kx_1 + 2p_1 + 4 \frac{\alpha}{\beta} + 2k \right) - \frac{\sqrt{2\lambda_2}}{18}
\] (7)

with
\[
\lambda_2 = 17(p_1 + k - kx_1)^2 - 24c \frac{\alpha}{\beta} + 18c^2 + 23 \left( \frac{\alpha}{\beta} \right)^2 - 12c(p_1 + k - kx_1) - 22 (p_1 + k - kx_1) \frac{\alpha}{\beta}
\]

Eq. (4–7) give the best response functions of the rivals with respect to prices and locations.

In a symmetric equilibrium where both firms charge the same mill price, \( p_1 = p_2 = p \), and locations are symmetric around the center, \( x_1 + x_2 = 0 \), the best response functions reduce to
\[
x_1 = \frac{1}{3k} \left( \frac{\alpha}{\beta} - p \right) - \frac{2}{3}; \quad x_2 = \frac{2}{3} - \frac{1}{3k} \left( \frac{\alpha}{\beta} - p \right);
\]
\(^2\)See Puu (2002, p. 8f.). A second solution for the mill prices does not fulfill the second-order condition, respectively.
\[
p = \frac{1}{10} \left[ 16k + 4 \frac{\alpha}{\beta} + 6c - 3 \sqrt{34k^2 - 8k \left( \frac{\alpha}{\beta} - c \right) + 4 \left( \frac{\alpha}{\beta} - c \right)^2} \right]
\]
and firm 1’s profit function may be rewritten as\(^3\).

\[
\Pi_1 = (p - c) \int_{-1}^{0} (\alpha - \beta p - \beta k|x - x_1|) \, dx
= (p - c) \left[ \alpha - \beta p - \frac{\beta k}{2} (2x_1^2 + 2x_1 + 1) \right]
\]

Using the best response functions gives the following expression for maximum profits in a symmetric duopoly:

\[
\Pi^d = \frac{\beta k^2}{250} \left[ (2\kappa^2 - 44\kappa + 377)\gamma - 92\kappa^2 + 482\kappa - 2194 \right]
\]

where \( \kappa = \frac{1}{k} \left( \frac{\alpha}{\beta} - c \right) > 0 \), \( \gamma = \sqrt{34 - 8\kappa + 4\kappa^2} \), and where the superscript \( d \) stands for ‘duopoly’. It turns out that all qualitative results hinge solely on the parameter \( \kappa \), which may be interpreted as the economic size of the market. An increase in the consumers valuation of the good, expressed by a higher prohibitive price \( \alpha/\beta \) has the same effect on it as lowering marginal costs \( c \) or a lower transportation cost parameter \( k \). The profits given by eq. (10) serve as a benchmark to find out whether the firms have an incentive to push the rival out of the market.

If \( \kappa \) is low (e.g. because the transportation cost rate is high), it may be that the two firms can locate sufficiently far apart that each is a local monopolist. This happens if the delivered price at the borders of each firm’s market area equals the prohibitive price \( \alpha/\beta \). For instance (firm 1’s delivered price at the market boundary \(-1\)):

\[
p + k(x_1 + 1) = \frac{\alpha}{\beta}
\]

Substituting \( x_1 \) and \( p \) by the symmetric best response functions yields \( \kappa = 3/4 \), i.e. if \( \kappa \leq 0.75 \), each firm is a local monopolist, which limits the scope of the model. In the limiting case \( \kappa = 3/4 \), firms locate at the quartiles, \( x_1 = -0.5; \ x_2 = 0.5 \), whereas in the duopoly case firms locate between the quartiles and the midpoint. This result complies with Hotelling’s intuition regarding the case of elastic demand, which is thus confirmed and formalized by Puu (2002). Hotelling himself was not able to derive a profit maximum in price and locations because with inelastic demand functions the profit function is quadratic in price and linear in location.

\(^3\)Of course, the profits of firm 2 must be the same.
3 Undercutting and stability

According to D’Aspremont et al. (1979), the equilibrium is stable if, for given locations, no firm may increase profits by undercutting the rivaling firm slightly at its site. It is evident that with increasing distance between the firms’ sites undercutting becomes less profitable. Put differently, only if the equilibrium locations of the firms are sufficiently far apart, stability in the described sense can be reached. Therefore, the ‘principle of minimum differentiation’ does not hold precisely because the equilibrium is stable.

Puu (2002) avoids to predetermine locations that are far apart as the only configuration that can generate stable equilibria by allowing the undercutting firm to select its location simultaneously. This is to say, while one firm sticks to the price and location corresponding with the formerly symmetric duopoly solution, the other firm can choose any price and location that pushes out the former firm. According to Puu (2002), the duopoly equilibrium is only stable if duopoly profits are higher than these undercutting profits. It is quite obvious that this concept is more ambitious than the one D’Aspremont et al. used because the additional choice parameter of the undercutting firm renders this option more advantageously.

We can restrict ourselves to analyze whether firm 1 has an incentive to push firm 2 out of the market, since firms are symmetric. Following Puu (2002), we denote the optimal undercutting mill price and location \( \tilde{p}_1 \) and \( \tilde{x}_1 \). Because the firm undercutts the rival’s price \( p \) by an arbitrarily small amount, and transportation costs are linear, the following relationship holds:

\[
\tilde{p}_1 = p - k(x_2 - \tilde{x}_1)
\]

Until here, our analysis essentially summarizes Puu (2002). But building the profit functions, Puu disregards that the undercutting firm’s optimal location may be so asymmetric that at the more distant end of the market the delivered price exceeds the prohibitive price \( \alpha/\beta \) (see Puu (2002, p. 14)). Ignoring this possibility may lower the calculated undercutting profits because at the most distant segments of the market demand would be taken as negative, while it is actually zero (see eq. 1). To find out when the restriction is binding, we first calculate, when the delivered price of the undercutting firm 1 at point \(-1\) reaches \( \alpha/\beta \).

\[
\frac{\alpha}{\beta} = \tilde{p}_1 + k(\tilde{x}_1 + 1)
\]

Using eq. (11), we get

\[
\overline{p}_1 = \frac{1}{2} \left( \frac{\alpha}{\beta} + p - k - kx_2 \right)
\]
If the optimal price is below this critical value, undercutting profits are

\[ \tilde{\Pi}_{1}^{u} = (\tilde{p}_{1} - c) \int_{-1}^{1} (\alpha - \beta \tilde{p}_{1} - \beta k|x - \tilde{x}_{1}|) \]  

where the superscript \( u \) stands for 'undercutting'. Substituting \( \tilde{p}_{1} \) by eq. (11), maximizing with respect to \( \tilde{x}_{1} \), and using the best response functions eq. (4–7) gives the optimum location and profits in the case of undercutting:

\[ \tilde{x}_{1} = \frac{1}{15} \left( \gamma - 3\kappa - 12 + \sqrt{36\gamma - 6\gamma\kappa + 34\kappa + 13\kappa^{2} - 17} \right) \]  

and

\[ \tilde{\Pi}_{1}^{u} = 2\frac{\beta k^{2}}{3375} \left[ 369\kappa^{2} - 29\gamma - 819\kappa - 2187 + 28\gamma\kappa - 63\kappa^{3} + 31\gamma\kappa^{2} + (36\gamma - 6\gamma\kappa + 34\kappa + 13\kappa^{2} - 17)^{3} \right] \]  

If, however, \( \tilde{p}_{1} > \overline{p}_{1} \), demand drops to zero at the internal point \( \overline{x} \), i.e. there are segments of the market that are not reached by the undercutting firm. From \( \alpha - \beta \tilde{p}_{1} - \beta k(\tilde{x}_{1} - \overline{x}) = 0 \) we get

\[ \overline{x} = \tilde{x}_{1} - \frac{1}{k} \left( \frac{\alpha}{\beta} - \tilde{p}_{1} \right) \]  

The profit function becomes

\[ \tilde{\Pi}_{1}^{u} = (\tilde{p}_{1} - c) \int_{\overline{x}}^{1} (\alpha - \beta \tilde{p}_{1} - \beta k|x - \tilde{x}_{1}|) \]  

Inserting eq. (16) for \( \overline{x} \) and eq. (11) for \( \tilde{p}_{1} \), using the best response functions eq. (4–7), and maximizing with respect to \( \tilde{x}_{1} \) yields

\[ \tilde{x}_{1} = \frac{1}{30} \left( 6\gamma + 2\kappa - 12 - \sqrt{498 - 216\kappa + 148\kappa^{2} + 24\gamma\kappa - 84\gamma} \right) \]  

and

\[ \tilde{\Pi}_{1}^{u} = \frac{\beta k^{2}}{13500} \left[ 6480\kappa^{2} + 9180\gamma - 14940\kappa + 360\gamma\kappa - 440\kappa^{3} + 360\gamma\kappa^{2} - 270\gamma^{3} + (498 - 216\kappa + 148\kappa^{2} + 24\gamma\kappa - 84\gamma)^{3} \right] \]  

For both cases, the corresponding mill price of the undercutting firm can be found by the relationship stated in eq. (11). We now address the question under which condition the critical price given by eq. (12) is reached, i.e. when eq. (13) or eq. (17) are valid, respectively. Setting equal the price corresponding to the optimal undercutting
location (14) and the critical price (12), inserting the previous results for \( p, \tilde{x}_1 \) and \( x_2 \), and solving for \( \kappa \) yields$^4$

\[
\kappa \approx 1.804
\]

Accordingly, if \( \kappa < \kappa \) the undercutting firm cannot reach some segments of the market in the neighborhood of the farther end (-1), and the results derived from profit function (17) are valid. If \( \kappa > \kappa \) the results derived from profit function (13) have to be used. If \( \kappa = \kappa \) these profits must coincide because the endogenously derived boundary of the market \( \bar{x} \) equals the exogenous end at -1.

Next, we compare the profits that respectively yield in the cases of undercutting, and in the case of oligopoly. Since we are mainly interested in whether undercutting profits are higher or lower than oligopoly profits, we divide undercutting profits (15) and (19) by duopoly profits (10), and see whether the resulting expressions are higher or lower than one. This has the advantage that the common factor \( \beta k^2 \) can be dropped, the ranking of the profits depends on \( \kappa \) only. Figure 1 illustrates how undercutting profits (15) and (19) fare relative to duopoly profits (10).

![Undercutting profits relative to duopoly profits](image)

Fig. 1: Undercutting profits relative to duopoly profits

Figure 1 reveals that undercutting profits with endogenous market boundary (eq. (19), dashed line) are always higher than those derived between -1 and 1 (eq. (15), solid line). Both are equal at \( \kappa \approx 1.804 \), where the exogenous and the endogenous market bound-

$^4$In addition, the third-order polynomial has one pair of complex roots.
ary coincide. The thicker curve respectively displays the firm’s profits. For \( \kappa > 1.804 \) the exogenous market boundary actually constrains the undercutting firm 1, profits are given by profit function (15). For \( \kappa < 1.804 \) the endogenous market boundary is closer to the firm than the exogenous boundary. Profit function (15) then integrates over market segments where the profit contribution is negative. Therefore, actual profits for \( \kappa \in [0.75, 1.804] \) are given by eq. (19). If \( \kappa \leq 0.75 \), two adjacent monopolies arise, so the limit of the undercutting profits function equals the ‘duopoly’ profits.

Taking into account that in the case of undercutting firms may not serve some fraction of the total market area, undercutting profits are always higher than duopoly profits (see fig. 1). Therefore, a stable duopoly equilibrium can never occur if the undercutting firm is allowed to relocate as in Puu (2002). This result is actually not surprising, and the following proposition extends it to cases where an unknown number of firms are non-symmetric, where demand may not be distributed evenly, and where the demand functions are not necessarily linear

**Proposition 1:** If

1. a given number of firms supplies a fixed market interval with a homogenous good,
2. competition actually takes place, i.e. demand at the boundary between any two firms’ market areas is strictly positive, and firms do not collude and
3. firms can choose price and location simultaneously and costlessly, taking their competitors prices and locations as given,

no stable equilibrium in prices and locations exists.

The proof of the proposition is straightforward and shall only be sketched here. In a stable equilibrium, no firm has an advantage from changing its location or its mill price. A firm relocating at the location of its closest competitor, and undercutting the price of this competitor by an epsilon would serve the entire market share of the competitor, and, if condition (2) is fulfilled, at least some fraction of its own former market share. This means that profits would be higher than before, unless its former position had been much better than that of the competitor. In the latter case, the competitor would have had an incentive to take the market of the considered firm, however. If we allow the undercutting firm to choose the optimal location (as Puu does), and this location is not the location of the competitor, profits must be even

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5Economides (1984, p. 349) points to a similar result in Economides (1982).
higher. This means that at least one firm has an incentive to undercut, and Puu’s concept of equilibrium is baseless.

Proposition 1 implies that, if one allows for simultaneous choice of locations and prices, most models in the tradition of Hotelling (1929) would become unstable, including D’Aspremont et al. (1979). If, however, the weaker conventional concept of equilibrium is used, where firms choose their location once and forever, Puu’s model is stable as well. We have calculated that, if firms must stay where they are, undercutting is a dominant strategy for all \( \kappa \geq 2.084 \) while duopoly is a Nash equilibrium in the interval \( \kappa \in (0.75, 2.084) \). Since it allows for elastic demand, and avoids the unintuitive assumption of convex (e.g. quadratic) transportation costs, it is superior to previous attempts to establish stable solutions to Hotelling’s location problem. Interestingly, it confirms Hotelling’s intuition that in the case of elastic demand “firms would gravitate closer together than placing themselves in the centres of their respective markets, but it would no longer lead to clustering in the same point” (Puu, 2002, p. 2).

In summary, replace Puu’s concept of stability by the established one, and everything’s fine? Not quite. Namely, criticism of the established way to tackle the issue of stability is only too appropriate. A two-stage game is solved recursively, where in the first stage the firms choose locations once and forever, and in the second stage prices are derived for given locations. Since the optimum location \( x \) depends on the equilibrium price \( p \) from the second stage of the game (see eq. (4) and (5)), the condition for a stable equilibrium can be written as

\[
\Pi^d(x(p^d), p^d) > \Pi^u(x(p^d), p^u)
\]

where, again, the superscripts \( d \) and \( u \) stand for 'duopoly' and 'undercutting', respectively. The fact that the location corresponds with the duopoly price in both cases means that, by checking stability of the duopoly solution, one implicitly assumes that the firm chooses the location knowing that it will not (be) undercut, i.e. that the equilibrium is stable. This informational assumption constitutes an inconsistency and should therefore be abandoned.

Unfortunately, as we have shown, Puu’s concept is no alternative. It is to be hoped that future research will provide more satisfactory concepts of equilibrium. One possible way to proceed would be to assume risk-averse firm owners, who are uncertain about the productivity of their firm relative to a competitor. This would constitute an incentive to locate farther apart, and possibly render undercutting unprofitable so that a stable equilibrium can emerge.
4 Summary

In this paper we show that Puu (2002) does not provide a stable solution to the location game, according to his own definition of stability. Only because the condition of non-negative demand is neglected, undercutting profits may be lower than duopoly profits. If the usual two-stage game is considered, where in the first stage (the location game) a location is chosen once and forever, and in the second stage an equilibrium in prices is derived, the equilibrium proves stable for a sizeable interval of the compound variable \( \kappa \), however. Even though this procedure is most common in analyzing Hotelling’s location problem, it is not satisfying because it exhibits an inconsistent informational structure. The search for a better concept of stability is imperative.

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