For CTRW subdiffusion in a semi-infinite domain, the $t^{-3/2}$ scaling thus changes to $\wp_\alpha(t) \simeq t^{-1-\alpha/2}$

In a finite domain the exponential decay of normal diffusion turns to the Mittag-Leffler function, and $\wp_\alpha(t) \simeq t^{-1-\alpha}$.

VIII. LANGEVIN FORMULATION OF CTRW & EXTENSIONS. [Fogedby, PRE (1994)]

CTRW: waiting time distribution $\psi(\tau)$ :: $\int_0^\infty \psi(\tau)\,d\tau = 1$

jump length distribution $\lambda(x)$ :: $\int_{-\infty}^\infty \lambda(x)\,dx = 1$

Position of the walker after s steps:

$$r(s) = \sum_{s'=0}^{s} x(s') \longrightarrow \int_0^s x(s')\,ds' \text{ in the continuum limit}$$

$$\Rightarrow \frac{dr}{ds} = x(s) \quad \text{Langevin eq. with random displacement } x$$

Time elapsed after s steps:

$$t(s) = \sum_{s'=0}^{s} \tau(s') \longrightarrow t(s) = \int_0^s \tau(s')\,ds'$$

$$\Rightarrow \frac{dt}{ds} = \tau(s)$$

With external forces: $\frac{dr}{ds} = F(r) + x(s)$

Sharp waiting times: $\psi(t) = \delta(\tau - \tau_0)$

$$\frac{dt}{ds} = \tau_0 \rightsquigarrow t(s) = \tau_0 s$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{\tau_0} F(r) + \frac{1}{\tau_0} x(t)$$

Otherwise we need to consider the coupled equations for r and t

Power-law forms for waiting times & jump lengths:

$$\lambda(x) \simeq |x|^{-1-\mu'}$$

$$\psi(t) \simeq t^{-1-\alpha'}$$

In the force-free case

$$P_1(r,s) = \langle \delta(r - r(s)) \rangle = \int \frac{dk}{2\pi} e^{ikr - |k|^\mu s} = s^{-1/\mu} G_1\left(\frac{r}{s^{1/\mu}}\right)$$

$$P_2(t,s) = \langle \delta(t - t(s)) \rangle = \int \frac{d\omega}{2\pi} e^{-i\omega t - (-i\omega)^\alpha s} = s^{-1/\alpha} G_2\left(\frac{t}{s^{1/\nu}}\right) \qquad (*)$$

For $\mu' > 2 \rightsquigarrow \mu = 2$, otherwise $\mu = \mu'$

$\alpha' > 1 \rightsquigarrow \alpha = 1$

Use scaling arguments:

$$\langle r^2(s) \rangle = \int P_1(r,s)\, r^2\, dr \simeq s^{2/\mu} \quad \text{because: } \int_{-a}^{a} \frac{r^2}{s^{1/\mu}} G_1\left(\frac{r}{s^{1/\mu}}\right) dr$$

$$= s^{2/\mu} \int_{-a}^{a} r'^2 G(r')\, dr' = f(a)\, s^{2/\mu}$$

$$\langle t(s) \rangle = \int t\, P_2(t,s)\, dt \simeq s^{1/\alpha}$$

$$\Rightarrow \langle r^2(t) \rangle \simeq (t^\alpha)^{2/\mu} = t^{2\alpha/\mu}$$

and for the PDF we obtain the scaling form

$$P(r,t) = t^{-\nu/\mu} G\left(\frac{r}{t^{\nu/\mu}}\right).$$ for $\nu = 1$, $\mu = 2$ we obtain the Gaussian by virtue of the CLT.

From (*) we see that $$\frac{\partial P_1(r,s)}{\partial s} = \frac{\partial^\mu}{\partial |x|^\mu} P_1(r,s) \,;\; \frac{\partial^\mu}{\partial |x|^\mu} = -\int \frac{dk}{2\pi} e^{ikr} |k|^\mu$$

and $\frac{\partial P_2(t,s)}{\partial s} = {}_0\mathcal{D}^\alpha P_2(t,s)$:. $\mathcal{D}^\alpha = -\int \frac{d\omega}{2\pi} e^{-i\omega t} (-i\omega)^\nu$

with drift due to the external force field:

$$\frac{\partial P_1}{\partial s} = \left(-\frac{\partial}{\partial x} F(r) + \frac{\partial^\mu}{\partial |x|^\mu} \right) P_1(r,s).$$

We eliminate the number of steps, s, through

$$P(r,t) = \int_0^\infty P_1(r,s) P_3(s,t)\, ds \quad \text{what is } P_3(s,t)? \qquad (**)$$

To this end, we use the known expression for free motion:

$$P(k,\omega) = \frac{(-i\omega)^{\alpha-1}}{(-i\omega)^\alpha + |k|^\mu}$$

From (**) one can then show that

$$P_3(s,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} (-i\omega)^{\alpha-1} e^{-(-i\omega)^\alpha s}$$

in analogy to our subordinator result in the Laplace variable.

Correlated CTRW processes - Langevin formulation. [Chechkin et al, PRE (2009)]

Idea: $\psi(\tau)$ governs IID random waiting times.

To introduce correlations between waiting times, change the way we generate the process time:

$$\frac{dt(s)}{ds} = \int_0^s M(s-s')\,\tau(s')\,ds' \quad \text{such that } M = \delta(s-s') \text{ reproduces Fogedby's eq.}$$

$$\frac{dx(s)}{ds} = \eta(s)$$

$$\Rightarrow t(s) = \int_0^s ds' \int_0^{s'} ds''\, M(s'-s'')\, \tau(s'') = \int_0^s \tau(s')\,\mu(s,s')\,ds' \;:.\; \mu(s,s') = \int_{s'}^s M(s''-s')\,ds''$$

In this case the characteristic function of the PDF becomes (analogous to the above approach):

$$P(k,s) = \exp\left(-|k|^{\alpha}\,\phi(s)\,\exp\left\{-\frac{i\pi\alpha}{2}\,\mathrm{sign}\,k\right\}\right) \quad \text{where } \alpha \text{ from } \psi(\tau) \simeq \tau^{-1-\alpha}$$

and $$\phi(s) = \int_0^s ds' \left[\int_{s'}^{s} M(s''-s')\,ds''\right]^{\alpha}$$

(i.) Exponential correlations: $M(s) = \Delta^{-1} e^{-s/\Delta}$

$$\phi(s) = \int_0^s \left[1-e^{-s'/\Delta}\right]^{\alpha} ds'$$

$$s \ll \Delta:\ \phi(s) \approx \int_0^s \left(\frac{s'}{\Delta}\right)^{\alpha} ds' = \frac{s^{1+\alpha}}{(1+\alpha)\Delta^{\alpha}} \Rightarrow \dots \Rightarrow \langle x^2(t)\rangle = K_\alpha\, t^{\alpha/(1+\alpha)} \quad t<\Delta$$

$$K_\alpha = 2\Delta^{\alpha/(1+\alpha)}\,(1+\alpha)^{1/(1+\alpha)}\,\frac{\Gamma(1/(1+\alpha))}{\alpha\Gamma(\alpha/(1+\alpha))}$$

$$s \gg \Delta:\ \langle x^2(t)\rangle = \frac{2t^{\alpha}}{\Gamma(1+\alpha)}$$

(ii.) Power-law correlations: $M(s) = \dfrac{s^{-\beta}}{\Gamma(1-\beta)}$, $0<\beta<1$

$$\rightsquigarrow \phi(s) = \frac{1}{[\Gamma(2-\beta)]^{\alpha}}\,\frac{s^{\nu}}{\nu} \quad \text{where} \quad \alpha(1-\beta)+1 \equiv \nu$$

$$\Rightarrow \dots \Rightarrow \langle x^2(t)\rangle = K_{\alpha,\beta}\, t^{\alpha/\nu} \;\therefore\; K_\alpha = 2\left[\Gamma(2-\beta)\right]^{\alpha/\nu}\,\frac{\nu^{1/\nu}\,\Gamma(1/\nu)}{\alpha\,\Gamma(\alpha/\nu)}$$

normal diffusion $\alpha=\beta=1 \rightsquigarrow \langle x^2(t)\rangle = 2\,t$

$\beta=1$: $\langle x^2(t)\rangle \simeq 2t^{\alpha}/\Gamma(1+\alpha)$ regular subdiffusion result

$\beta<1$: $\langle x^2(t)\rangle \simeq t^{\alpha/\nu}$ slower than t^{α}

$\beta\to 0$: $\langle x^2(t)\rangle \simeq t^{\alpha/(1+\alpha)}$ $\therefore$ $\alpha/(1+\alpha) \in (0, 1/2)$.

PDF?, first passage?, force field?