# Lévy Stable Probability Laws, Lévy Flights, Space-Fractional Diffusion and Kinetic Equations (Two lectures)

(Paul Pierre Lévy, 1886-1971)
(Stable probability laws, alpha-stable probability laws/distributions)
Math: = alpha-stable Levy random processes, Levy motion.

Mandelbrot: Lévy flights (LFs) = non-Brownian random walks with long jumps (physics, nature, finance)

We need a probabilistic background which goes beyond the scope of "traditional" courses for physicists on probability theory

## I. Stable probability laws (stable distributions) and generalized central limit theorem

**# 1 Definition. Stability = invariance under summation.** $X_1, X_2, X_3, \quad , X$ i.i.d. (independent identically distributed) r.v. (random variables) possess a stable distribution if $\exists\, c_n > 0$ and $d_n$ such that

$$X_1 + X_2 + \ . \ . \ . \ + X_n = \sum_{i=1}^{n} X_i \stackrel{d}{=} c_n X + d_n \quad . \tag{1}$$

$\stackrel{d}{=}$ equal in probability

$c_n$ scaling factor

Theorem. $\boldsymbol{c_n = n^{1/a}}$ **, $0 < \alpha \leq 2$, $\alpha$ index of stability (math), or Lévy index (phys).**

**Alternative definition**. $X1$, $X2$, and $X$ i.i.d. r.v. possess stable distribution if for $\forall$ A > 0, B > 0 $\exists$ C > 0, D such that

$$AX_1 + BX_2 \stackrel{d}{=} CX + D \tag{2}$$

Then $\exists\ \alpha \in$ (0,2] such that

$$C^\alpha = A^\alpha + B^\alpha \quad . \tag{3}$$

**Noteworthy examples:**

**1.** $\alpha = 2$ (Gaussian distribution)

Probability density function (PDF) $p_G(x;\sigma,\mu) = \dfrac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$ , $\displaystyle\int_{-\infty}^{\infty} dx\, p_G(x;\sigma,\mu) = 1$

$\mu$ mean, $\sigma^2$ variance.

**2.** $\alpha = 1$ (Cauchy – Lorenz probability law)

PDF $p_C(x;\gamma,\delta)=\frac{1}{\pi}\frac{\gamma}{(x-\delta)^2+\gamma^2}$ , $\int\limits_{-\infty}^{\infty} dx\, p_C(x;\gamma,\delta)=1$ .

These two distributions belong to the class of stable distributions, but they are very different.

Gauss: all the moments are finite.

Cauchy: Asymptotics. $p_C \propto \frac{1}{x^2}$

Cauchy: no integer moments, only fractional moments of the order $q < 1$ are finite:

$$\left\langle |x|^q \right\rangle = \int\limits_{-\infty}^{\infty} dx|x|^q\, p_C(x;\gamma,\delta) \sim \int\limits_{0}^{\infty} dx\, x^q x^{-2} < \infty \quad if \quad q<1 \quad .$$

## # 2 Characteristic function (CF) of stable distribution

$$\text{CF} \quad p(k)=\int\limits_{-\infty}^{\infty} dx p(x)e^{ikx}=\left\langle e^{ikx}\right\rangle \quad , \quad p(x)=\int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}p(k)e^{-ikx} \quad . \tag{4}$$

$$\left\langle e^{ik(X1+X2+...+Xn}\right\rangle=\left\langle e^{ikX1}\right\rangle\left\langle e^{ikX2}\right\rangle...\left\langle e^{ikXn}\right\rangle=\left\langle e^{ikXcn}\right\rangle e^{ikd_n} \quad .$$

**<u>Functional equation for CF</u>** $p(k)$ : $\left[p(k)\right]^n = p(c_n k)e^{id_n k}$ .

Solution (Lévy; Khintchine; Gnedenko, Kolmogorov):

$$p_{\alpha,\beta}(k;\mu,\sigma)=\exp\left[i\gamma k-\sigma^{\alpha}\,|\,k\,|^{\alpha}\,(1-i\beta\omega(k,\alpha))\right] \quad , \tag{5}$$

where

$$\omega(k,\alpha)=\begin{cases}\tan(\pi\alpha/2) & if \quad \alpha\neq 1, \quad 0<\alpha<2\\ -(2/\pi)\ln|\,k\,| & if \quad \alpha=1\end{cases} \quad .$$

4-parametric: $0 < \alpha \leq 2$ (Levy index), $-1 \leq \beta \leq 1$ (skewness parameter), $\gamma > 0$ (scale parameter), $\delta$ real (shift parameter).

## # 3 Properties of probability densities.

**(1)** $\alpha$ and $\beta$ are important ! $\gamma$ and $\delta$ are less important ($\gamma$ and $\delta$ can be eliminated by proper shift and scale transformation:

$$p_{\alpha,\beta}(x;\gamma,\sigma)=\frac{1}{\sigma}p_{\alpha,\beta}\left(\frac{x-\gamma}{\sigma};0,1\right) .$$

Notation: $p_{\alpha,\beta}(x;0,1)\equiv p_{\alpha,\beta}(x)$

**(2)** Symmetry property:

$$p_{\alpha,-\beta}(x) = p_{\alpha,\beta}(-x) \quad .$$

**(3)** Symmetric stable laws: $\beta = 0$ $p_{\alpha,0}(x) = p_{\alpha,0}(-x)$ (Next lecture: play important role in random walk theory).
CF for symmetric stable law:

$$p_{\alpha,0}(k) \equiv p_\alpha(k) = \exp\left(-|k|^\alpha\right) \quad .$$

$$p_\alpha(k) \simeq 1 - |k|^\alpha + \frac{1}{2}|k|^{2\alpha} - \ldots \quad k \to 0 \quad .$$

The first non-analytical term is responsible for asymptotics of the PDF at $x \to \infty$:

$$p_\alpha(x) \approx \frac{C(\alpha)}{|x|^{1+\alpha}} \quad , \quad x \to \pm\infty \quad .$$

[Here: digression on the derivation of C(α) with the use of improper integral].

$$C(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha) \quad .$$

**Examples. In terms of elementary functions:**
Example 1. Gauss <To derive from CF>

$$p_2(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4} \quad .$$

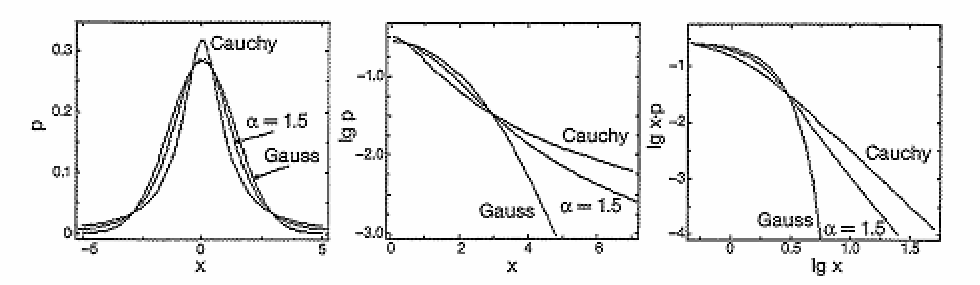
Example 2. Cauchy <To derive from the CF>

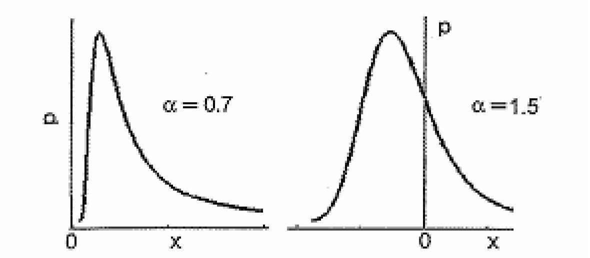
$$p_1(x) = \frac{1}{\pi(1+x^2)} \quad .$$

**(4)** Extremal stable laws $\beta = \pm 1$. Important subclass $0 < \alpha < 1$. Next lecture: play important role in random walk theory as distributions of waiting times.

In terms of elementary function: only for $\alpha = 1/2$ , $\beta = 1$.
Example 3. Lévy-Smirnov.

$$p_{1/2,1}(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi x^3}} e^{-1/2x} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases} \quad .$$




B. Gnedenko, A. Kolmogorov, Limit Distributions for Sums of Independent Random Variables (1949), two citations:
***<u>"The prophecy"</u>***: "All these distribution laws, called stable, deserve the most serious attention. It is probable that the scope of applied problems in which they play an essential role will become in due course rather wide."
***<u>"The guidance for those who write books on statistical physics"</u>***: "… "normal" convergence to abnormal (that is, different from Gaussian –A. Ch.) stable laws … , undoubtedly, has to be considered in every large textbook, which intends to equip well enough the scientist in the field of statistical physics."

## #4 Generalized central limit theorem GCLT

***<u>Generalized Central Limit Theorem</u>: stable distributions are the limit ones for distributions of sums of random variables (has domains of attraction) ⇒ appear, when evolution of the system or the result of experiment is determined by the sum of a large number of random quantities***

$X_1$, $X_2$, $X_3$, ….$X_n$ i.i.d.r.v. PDF $f(x)$: what is the distribution of their sum at $n \to \infty$ ?

**- Gauss, $\alpha = 2$: attracts $f(x)$ with <u>finite</u> variance**

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x) dx < \infty$$

**- Lévy, $0 < \alpha < 2$: attract $f(x)$ with <u>infinite</u> variance and the <u>same</u> asymptotic behavior**

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x) dx = \infty \quad , \quad f(x) \sim x^{-1-\alpha} \, , \quad 0 < \alpha < 2$$

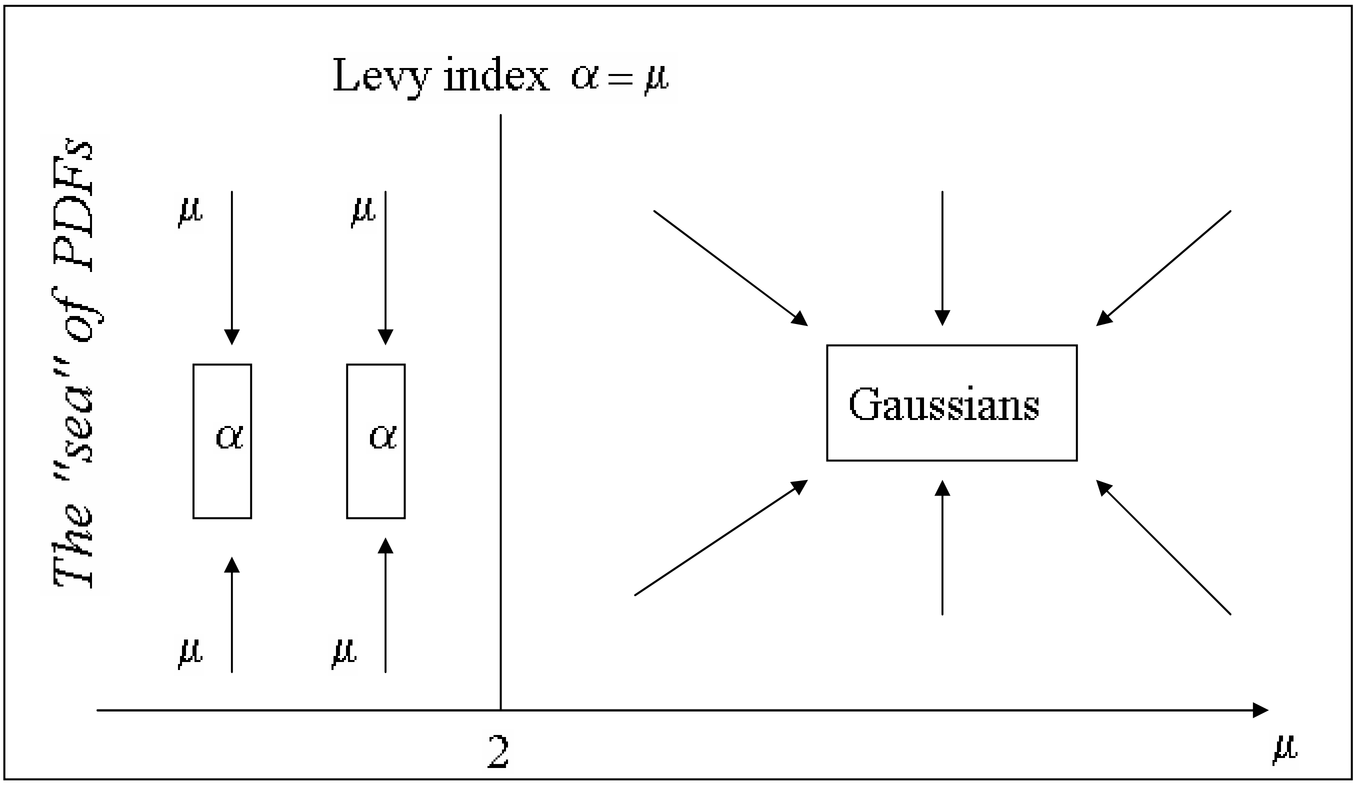
**More precisely: normal convergence to <u>symmetric</u> stable laws**

$$X_1, \ \ldots \ X_n - i.i.d.r.v. \ \ f(x) \sim \frac{1}{|x|^{1+\mu}}, \ \ x \to \infty$$

$$0 < \mu < 2$$

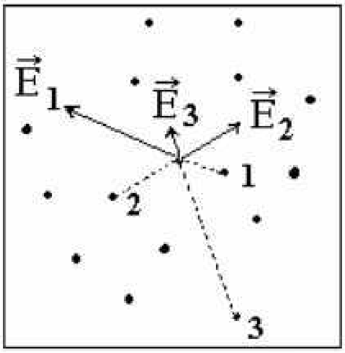
$$S_n = \frac{X_1 + \ \ldots \ X_n}{n^{1/\mu}} : \varphi_n(x)$$

$$\varphi_n(x) \underset{n \to \infty}{\to} p_\alpha(x) \sim \frac{1}{|x|^{1+\alpha}}, \ \ x \to \infty, \quad \alpha = \mu$$



## Electric Field Distribution in an Ionized Gas (Holtzmark 1919)

*Application:* **spectral lines broadening in plasmas**




$$\vec{E}_i = \frac{e\vec{r}_i}{r_i^3} \qquad n = \frac{N}{V} = const$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n$$

$$W\left(\vec{E}\right) = \int_{-\infty}^{\infty} \frac{d\vec{k}}{(2\pi)^3} \widehat{W}\left(\vec{k}\right) e^{-i\vec{k}\vec{E}}$$

$$\widehat{W}\left(\vec{k}\right) = e^{-a\left|\vec{k}\right|^{3/2}} \qquad \widehat{W}\left(\left|\vec{E}\right|\right) \sim \frac{1}{\left|\vec{E}\right|^{5/2}}$$

3 – dimensional
symmetric
stable
distribution
$\alpha = 3/2$

Also:

- gravity field of stars ($\alpha$ = 3/2)
- electric fields of dipoles (= 1)
- velocity field of point vortices in fully developed turbulence (=3/2) ……

## II. Continuous time random walk and space fractional diffusion equation

**# 1 Basics of CTRW**

Return to random walk (RM Lecture #2) : $P(x,t)$ PDF to find a particle at position $x$ at time $t$.

-- random jumps , PDF $\lambda(\xi)$, $\lambda(k)=F\{\lambda(\xi)\}=\int_{-\infty}^{\infty} dx\, \lambda(x)e^{ikx}$

-- random waiting times, PDF $\psi(\tau)$, $\psi(u)=L\{\psi(\tau)\}=\int_0^{\infty} d\tau\, \psi(\tau)e^{-u\tau}$

-- **Montroll-Weiss** equation (RM lecture #2)

$$P(k,u)=\frac{1-\psi(u)}{u}\frac{1}{1-\lambda(k)\psi(u)} \quad . \tag{2.1}$$

-- No anomaly in waiting time PDF (RM lectures 3-4)

$$\langle\tau\rangle<\infty \Rightarrow \quad \psi(u)\simeq 1-\langle\tau\rangle u+O(u^{\mu}) \quad , \quad u\to 0\,, \quad \mu>1 \ , \tag{2.2}$$

$$\langle\tau\rangle=\int_0^{\infty} d\tau\, \tau\psi(\tau)=-\left(\frac{d}{du}\int_0^{\infty} d\tau\, \psi(\tau)e^{-u\tau}\right)\Bigg|_{u=0}=-\frac{d\psi(u)}{du}\Bigg|_{u=0} \quad .$$

**# 2 Two cases for jump PDF λ(ξ).**

**Case 1.** Jump PDF belongs to domain of attraction of the Gaussian law.

$$\langle\xi^2\rangle=\int_{-\infty}^{\infty} d\xi\, \xi^2\lambda(\xi)<\infty \quad \Rightarrow \quad \lambda(k)\simeq 1-\frac{\langle\xi^2\rangle}{2}k^2+O(k^{\mu}) \quad , \quad k\to 0, \mu>2 \ .$$

$$\langle\xi^2\rangle=-\left(\frac{d^2}{dk^2}\int_{-\infty}^{\infty} dx\, \lambda(x)e^{ikx}\right)\Bigg|_{k=0}=-\left(\frac{d^2\lambda(k)}{dk^2}\right)\Bigg|_{k=0} \quad .$$

**MW** equation gives

$$P(k,u)=....=\frac{1}{u+Dk^2} \quad , \quad D=\frac{\langle\xi^2\rangle}{2\langle\tau\rangle} \quad . \tag{2.3}$$

$$uP(k,u)-1=-Dk^2P(k,u) \quad .$$

Now, recalling that

$$L\left\{\frac{d}{dt}\phi(t)\right\}=u\phi(u)-\phi(t=0) \quad , \quad F\left\{\frac{d^2}{dx^2}\varphi(x)\right\}=-k^2\varphi(k) \quad ,$$

we can immediately make an inverse Fourier-Laplace transform of (2.3) to get normal diffusion equation

$$\frac{\partial}{\partial t}P(x,t)=D\frac{\partial^2}{\partial x^2}P(x,t) \quad , \quad P(x,t=0)=\delta(x)\,, -\infty<x<\infty\,, \quad D=\frac{\langle\xi^2\rangle}{2\langle\tau\rangle} \quad . \tag{2.4}$$

- Solution:

$$P(x,t)=\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}} \quad . \tag{2.5}$$

- mean square displacement

$$\langle x^2\rangle=\int_{-\infty}^{\infty}dx\, x^2P(x,t)=2Dt\propto t^1 \quad : \qquad \textbf{normal diffusion}$$

**Case 2**

Jump PDF belongs to domain of attraction of the stable law with $0<\alpha<2$.

That means

$$\langle\xi^2\rangle=\int_{-\infty}^{\infty}d\xi\,\xi^2\lambda(\xi)=\infty \quad , \quad \lambda(\xi)\propto\frac{\sigma^\alpha}{|x|^{1+\alpha}}, \quad 0<\alpha<2\,, \quad |x|\to\infty$$

$$\Rightarrow \quad \lambda(k)\simeq 1-\sigma^\alpha|k|^\alpha+\ldots \quad , \quad k\to 0 \tag{2.6}$$

(Remind digression on the improper integral).

Plug **(2.6)** and **(2.2)** into the **MW** equation (2.1):

$$P(k,u)=\frac{\langle\tau\rangle}{\langle\tau\rangle u+\sigma^\alpha|k|^\alpha}=\frac{1}{u+D|k|^\alpha} \quad , \quad D=\frac{\sigma^\alpha}{\langle\tau\rangle} \quad .$$

or

$$uP(k,u)-1=-D|k|^\alpha\,P(k,u) \quad . \tag{2.7}$$

Rewrite (2.7) as

$$\frac{\partial}{\partial t}P(x,t)=D\frac{\partial^\alpha}{\partial|x|^\alpha}P(x,t) \quad , \quad P(x,t=0)=\delta(x)\,, \quad 0<\alpha<2$$

**# 3 Riesz fractional derivative**

$$\phi(x) \div \phi(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}\phi(x) \quad , \quad \phi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx}\phi(k)$$

$$\frac{df}{dx} \div -ikf(k) \qquad\qquad \frac{d^2 f}{dx^2} \div (-ik)^2 f(k) = -k^2 f(k)$$

$$\frac{d^\alpha f}{d|x|^\alpha} \equiv -(-\Delta)^{\alpha/2} \div -|k|^\alpha f(k)$$

$$\frac{d^2 f}{d|x|^2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left(-k^2\right) f(k) e^{-ikx} = \frac{d^2 f}{dx^2}$$

Example.

$$\phi(x) = \frac{1}{1+x^2}$$

$$\frac{d^\alpha \phi}{d|x|^\alpha} = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(-ikx)|k|^\alpha \phi(k) = -\frac{\Gamma(\alpha+1)}{\pi\left(1+x^2\right)^{(\alpha+1)/2}} \cos\left[\left(\alpha+1\right)\arctan x\right]$$

**Conclusion: space-fractional diffusion equations emerge as a long time-space limit of a random walk with long random jumps whose PDF $\lambda(\xi)$ belongs to domain of attraction of stable law:**

$$\lambda(\xi) \propto \frac{\sigma^\alpha}{|\xi|^{1+\alpha}} \quad 0<\alpha<2 \quad , \quad \left\langle \xi^2 \right\rangle = \infty$$

**(Section II "CTRW and Space-Fractional Diffusion Equation" continued)**

## # 4 Space-fractional diffusion equation for Lévy stable jump PDF

Jumps in discrete time:

$$X(n)=\sum_{j=1}^{n}\xi_j \quad , \quad \lambda(\xi)\div\lambda(k)=\left\langle e^{ik\xi_j}\right\rangle=\exp\left(-\sigma^\alpha|k|^\alpha\right)$$

$$0<\alpha\le 2 \qquad P_X(x,n)=?$$

$$P_X(k,n)=\left\langle e^{ikX(n)}\right\rangle=\left\langle e^{ik\sum_{j=1}^{n}\xi_j}\right\rangle=\left\langle e^{ik\xi_j}\right\rangle^n=e^{-n\sigma^\alpha|k|^\alpha}$$

Go to the continuous time by $t=n\Delta t$

$$P_X(k,t)=e^{-tD|k|^\alpha} \quad , \quad D=\frac{\sigma^\alpha}{\Delta t}$$

Differentiate:

$$\frac{d^\alpha\phi}{d|x|^\alpha}\div -|k|^\alpha\phi(k)$$

$$\frac{\partial}{\partial t}P(k,t)=-D|k|^\alpha P(k,t)$$

$$\frac{\partial}{\partial t}P(x,t)=D\frac{\partial^\alpha}{\partial|x|^\alpha}P(x,t)$$

## # 5 Solution of space-fractional diffusion equation, *q*-th moments and superdiffusion

Go back to Fourier space:

$$P(k,t)=\exp(-D|k|^\alpha t)$$

Asymptotics by using the same trick as shown in Section I:

$$P(x,t)\propto\frac{Dt}{|x|^{1+\alpha}}$$

Reminder: $\left\langle x^2\right\rangle=\infty$ (consider the integrand), but $\left\langle |x|^q\right\rangle<\infty \quad , \quad q<\alpha$ .

Time behavior of the *q*-th moment:

$$\left\langle |x|^q \right\rangle = \int_{-\infty}^{\infty} dx\, |x|^q P(x,t) = 2\int_0^{\infty} dx\, x^q \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ikx - D|k|^\alpha t} = \left\langle k \to k' = k(Dt)^{1/\alpha} \right\rangle =$$

$$= 2\int_0^{\infty} dx\, x^q \frac{1}{(Dt)^{1/\alpha}} \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, e^{-ik'x/(Dt)^{1/\alpha} - |k'|^\alpha} = \left\langle x \to x' = \frac{x}{(Dt)^{1/\alpha}} \right\rangle =$$

$$= 2(Dt)^{q/\alpha} \int_0^{\infty} dx'\, x'^q \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, e^{-ik'x' - |k'|^\alpha} \quad .$$

Effective second moment:

$$M_2(t;\alpha,q) = \left\langle |x|^q \right\rangle^{2/q} \propto (Dt)^{2/\alpha} \quad , \quad \frac{2}{\alpha} > 1 \quad , \quad 0 < \alpha < 2 \quad .$$

## Superdiffusion phenomenon

- **Examples of LFs:**
  - **Optics photons,**
  - **Levy glass**
  - **Strange diffusion along the DNA**
  - **Scaling law of human travel**

**Imprisonment of Resonant Radiation in Gases: Photon Trajectories as Lévy Flights**

♦ **Compton (1922): transfer of excitation as a type of Brownian motion ⇒ diffusion-type equation for the density of excited atoms**

$$\partial n / \partial t = D \nabla^2 n$$

Essential assumption: $D = \lambda^2 / 3\tau$ ⟺ $p(r) \propto \exp(-r/\lambda)$

**Mean free path**

**BUT !** Kenty (1932), Holstein (1947): frequency dependent absorption

$$p(r) \propto \frac{1}{r^2 (\ln r)^{1/2}}$$ or $$p(r) \propto \frac{1}{r^{3/2}}$$ ⇒ $$\lambda = \infty$$

**Lévy flights of photons in hot atomic vapours**

(Mercadier et al. Nature Physics 2009)

- **Measuring the step size distribution of photons**

- **Under Gaussian assumptions of emission and absorption spectra (Doppler broadening)**

$$P(r) \approx \frac{1}{r^2 \sqrt{\ln(r/l_0)}}$$

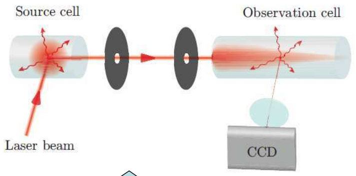



Figure: Double cell configuration to measure the step size PDF

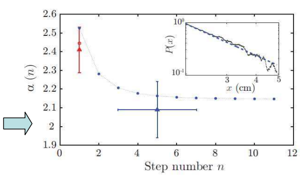
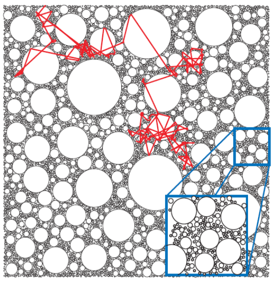
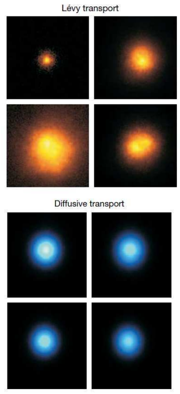



Figure: power law exponent α of the step size distribution vs number $n$ of scattering events (multiply scattering)


## A Lévy Flight for Light
(P. Barthelemy et al, Nature 2008)






• Ideal experimental system to study Lévy flights in a controlled way

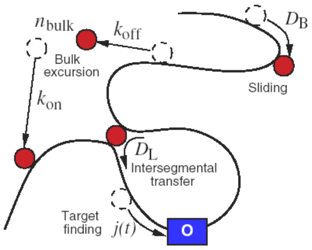
• Precisely chosen distribution of glass microspheres of different diameters d
P(d) ~ $d^{-(2+\alpha)}$

• New optical material in which light performs a Lévy flight – Lévy glass

Spatial distribution of the transmitted intensity

## "Paradoxical" Diffusion in Chemical Space

Protein diffusion to next neighbor sites on folding DNA (deoxyribonucleic acid)




• Motion of binding proteins or enzymes along DNA: detach to a volume → reattach before reaching the target (Berg – von Hippel model)

• Intersegmental jumps permitted at chain contact points due to polymer looping

• Contoured length |x| stored in a loop between contact points

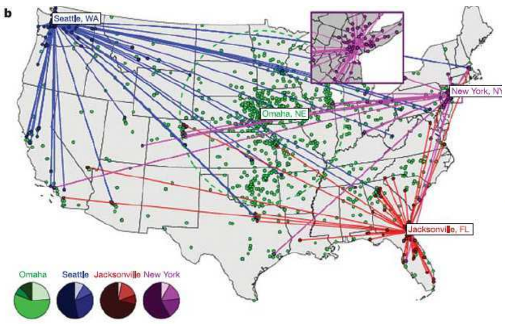
$$\lambda(x) \sim |x|^{-1-\alpha}$$
$$\alpha = 1/2$$

Optimal Target Search on a Fast-Folding Polymer Chain (Lomholt et al.2005)

$$\frac{\partial}{\partial t} n(x,t) = \left( D_B \frac{\partial^2}{\partial x^2} + D_L \frac{\partial^\alpha}{\partial |x|^\alpha} - k_{off} \right) n(x,t) + k_{on} n_{bulk} - j(t)\delta(x)$$

## Scaling laws of human travel, $\mu \approx 2$
(Brockmann, Hufnagel, Geisel, Nature 2006)

***Money as a proxy for human travel: trajectories of banknotes originated from 4 places $\Rightarrow$ reminiscent of Lévy flights in the US currency tracking project***

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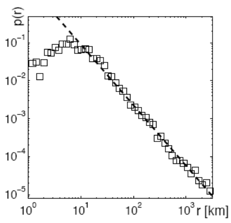



$r = |\vec{x}_2 - \vec{x}_1|$ Geographical displacement *r* between two report location of a bank note

$$\frac{\partial^\beta}{\partial t^\beta} W(t,r) = D \frac{\partial^\alpha}{\partial |r|^\alpha} W(t,r), \quad \alpha \approx \beta \approx 0.6, \mu = 2\beta/\alpha$$

## III. Lévy Flights, Lévy motion / Alpha stable Lévy processes (symmetric case)

The same phenomenon from the random processes theory point of view.

### # 1 <u>Definitions, terminology, examples</u>.

**(1)** $L_\alpha(t)$: non-stationary Markov process with stationary independent increments distributed with the alpha-stable law

$$p_\alpha(k,\tau) = \left\langle \exp\left[ik\left(L_\alpha(t+\tau) - L_\alpha(t)\right]\right\rangle = \exp\left(-D\,|\,k\,|^\alpha\,\tau\right)$$

**(2) Stationarity of increments:**

$$L_\alpha(t_1+\tau) - L_\alpha(t_2+\tau) \overset{d}{=} L_\alpha(t_1) - L_\alpha(t_2)$$

**(3) Physically: increments = jumps**

**(4) White Lévy noise = Sequence of increments**

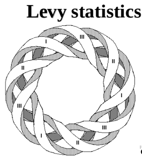
**(5) Physically : description of processes with large outliers, far from equilibrium.**

<u>Examples of the Lévy noises include</u> :

- economic stock prices and current exchange rates (1963)
- radio frequency electromagnetic noise
- underwater acoustic noise
- noise in telephone networks
- biomedical signals
- stochastic climate dynamics
- turbulence in the edge plasmas of thermonuclear devices (Uragan 2M,ADITYA,2003; Heliotron J,2005...)

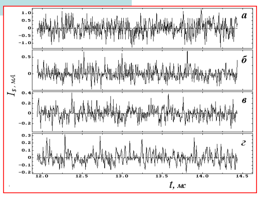
«Levy Turbulence» in boundary plasma of thermonuclear devices

Levy statistics of plasma fluctuations:



- •• Uragan-3M
- •• ADITYA
- •• L-2M
- •• LHD, TJ-II
- •• Heliotron J

acter of fluctuations in other devices (DIII-D, TCABR, …)



"*Levy turbulence*" is a widely spread phenomenon ⇒ **Models are strongly needed !**

**Motion of a test particle in the Guiding Center Approximation :**

$\delta\vec{E}(\vec{r},t)$ : Levy random field

$$\frac{d\vec{r}}{dt} = \frac{\vec{E}_0(\vec{r})\times\vec{B}_0}{B_0^2} + \frac{\delta\vec{E}(\vec{r},t)\times\vec{B}_0}{B_0^2}$$

## # 2 First passage and arrival properties of LFs

Two random quantities: first passage time and first passage overshooting (leapover) $l$: the motion starts at $x_0 > 0$ the boundary is at $x = 0$

$$p(t;x_0) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_0^\infty dx f(x,t)$$

Probability to leave the interval (0, ∞) within the time from t till t+dt.

**(1) Universality of the First passage time PDF / "Sparre Andersen Universality" .**

$$p(t) \propto t^{-3/2} \quad , 0 < \alpha \le 2$$

**(2) Gaussian case** $f(x,t) = W(x - x_0, t) - W(x + x_0, t)$

$$p(t;x_0) = \frac{x_0}{\sqrt{4\pi D t^3}} e^{-x_0^2/(4Dt)} \propto t^{-3/2}$$

$$W(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

$$\langle t \rangle = \int_0^\infty dt\, t\, p(t;x_0) = \infty$$

**(3) LFs:**

$$f(x,t) = f_{im}(x,t) = W(x - x_0, t) - W(x + x_0, t) \quad ?$$

$$W(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(-ikx) \exp\left(-D|k|^\alpha t\right)$$

$$p_{im}(t;x_0) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_0^\infty dx f_{im}(x,t)$$

$$p_{im}(t) \propto t^{-1-1/\alpha} \ne t^{-3/2}$$

**Method of images breaks down for Lévy flights due to their non-local nature, which manifests itself in the non-zero leapover**

**(4) Probability of first arrival time (Problem #3 from the assignment sheet).**

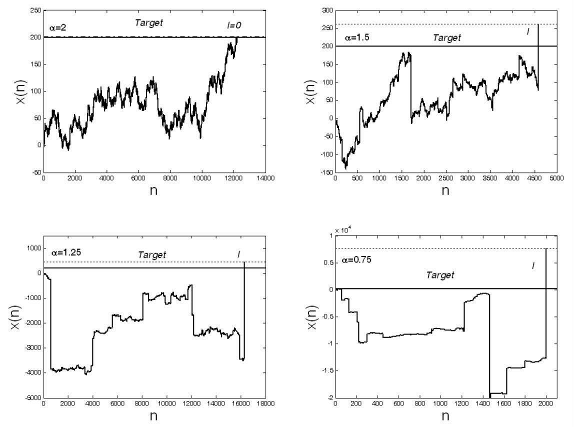
$$\frac{\partial f}{\partial t} = D \frac{\partial^\alpha f}{\partial |x|^\alpha} - p_{fa}(t)\delta(x)$$

$$f(x,t=0)=\delta(x-x_0)\,,\,x_0>0\,,\quad f(x=0,t)=0$$

$$p_{fa}(t)\propto t^{-2+1/\alpha}\neq t^{-3/2}$$

**First arrival $\neq$ first passage for LFs !**

**(5) First passage overshooting (leapover)**




PDF of overshooting:

$$f(l)\propto\frac{1}{l^{1+\alpha/2}}$$

Reminder. For LFs $\quad p(x)\propto\dfrac{1}{|x|^{1+\alpha}}\quad$ .

**The PDF of the first overshooting decays slower than the PDF of LFs which produce this overshooting !**

## IV. Langevin Equation with Lévy noise and space-fractional Fokker-Planck Equation

**# 1 Langevin equations : 2nd Newton's law**

$$m\frac{d\mathrm{v}}{dt}=-\frac{dU}{dx}-\gamma m\mathrm{v}+\xi_{\alpha}(t)\quad,\quad\frac{dx}{dt}=\mathrm{v}$$

$\xi_\alpha(t)$ : white Lévy noise

- overdamped approximation: neglect inertia.

**# 2 Fokker-Planck equation**

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial x}\left(\frac{dU}{dx}f\right)+D\frac{\partial^{\alpha}f}{\partial|x|^{\alpha}}$$

**$\alpha = 2$ : "normal" Fokker-Planck equation**

**# 3 Reminder. Boltzmann distribution**

$$\frac{d}{dx}\left(\frac{dU}{dx}f\right)+D\frac{d^2 f}{dx^2}=0$$

$$f(x)=C\exp\left(-\frac{U(x)}{D}\right)\quad,\quad\int_{-\infty}^{\infty}dxf(x)=1$$

**# 4 Stationary solution for LFs in Harmonic Potential**

$$U(x)=\frac{x^2}{2}\ ,\qquad\qquad \frac{d^{\alpha}\phi}{d|x|^{\alpha}}\ \div\ -|k|^{\alpha}\phi(k)$$

**<To derive the equation for the CF>**

$$f(k)=\exp\left(-\frac{|k|^{\alpha}}{\alpha}\right)$$

$$f(|x|\to\infty)\approx\frac{C}{|x|^{1+\alpha}},\quad C=\frac{\sin(\pi\alpha/2)\Gamma(\alpha)}{\pi},\qquad\left\langle x^2\right\rangle=\int_{-\infty}^{\infty}dx\,x^2 f(x)=\infty$$

**Properties of stationary PDF: unimodality and slowly decaying asymptotics**

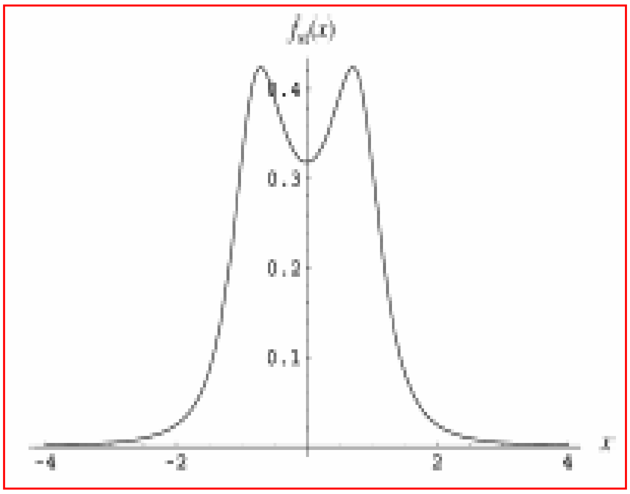
## # 5 Stationary Solution of LFs in Quartic Potential

$$\frac{dx}{dt} = -x^3 + Y_\alpha(t)$$

$$\frac{d}{dx}\left(x^3 f\right) + \frac{d^\alpha f}{d|x|^\alpha} = 0$$

$$\frac{d^3 f}{dk^3} = \mathrm{sgn}(k)|k|^{\alpha-1} f(k)$$

$$f(x) = \frac{1}{\pi(1 - x^2 + x^4)} \quad , \quad \alpha = 1$$




$$x_{\min} = 0 \qquad x_{\max} = \pm 1/\sqrt{2}$$

$$f(x) \propto x^{-4} \qquad \left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} dx \;\; x^2 f(x) = 1$$

**Confined LFs: finite variance, out of the domain of attraction of stable laws**