For TA MSD we find:

$$\left\langle \overline{\delta^2(\Delta; t_a, T)} \right\rangle = \frac{1}{T-\Delta} \int_{t_a}^{t_a+T-\Delta} \left(x(t+\Delta) - x(t) \right)^2 dt$$

$$= \frac{\Lambda_\alpha(t_a/T)}{\Gamma(1+\alpha)} \frac{g(\Delta)}{T^{1-\alpha}}$$

$g(\Delta) = \Delta$ for free motion, $g(\Delta) \simeq \Delta^{1-\alpha}$ under confinement

universal prefactor $\Lambda_\alpha(z) = (1+z)^\alpha - z^\alpha$ for entire range of Δ!

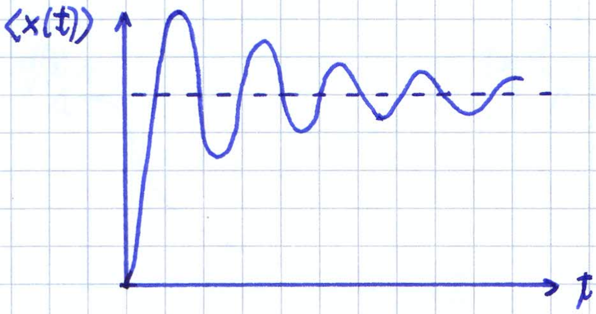
"Death of linear response":

In presence of a time-dependent external force $F(t)$, the fractional Fokker-Planck equation becomes (Sokolov & Klafter, PRL 97, 140602 (2006)):

$$\frac{\partial}{\partial t} P(x,t) = \left(-\frac{F(t)}{m\eta_\alpha} \frac{\partial}{\partial x} + K_\alpha \frac{\partial^2}{\partial x^2} \right) {}_0D_t^{1-\alpha} P(x,t).$$

Consider an harmonic forcing of the form $F(t) = F_0 \sin \omega t$.
The first moment becomes

$$\langle x(t) \rangle = \frac{F_0}{m\eta_\alpha} \sqrt{2\pi}\, S\left(\sqrt{\frac{2t}{\pi}}\right)$$ in terms of the Fresnel integral $S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$



Thus the particle experiences an initial shift. Due to longer & longer waiting times, it shows oscillations of decreasing amplitude.

VII. First Passage Phenomena.

In many cases it is of interest when a stochastic variable crosses a given amount for the first time. Examples include chemical reactions, stock market dynamics, or search for a target by animals.

We determine the PDF of first arrival, $\wp(t)$, from the following boundary condition: $P(x_t, t) = 0$, where x_t is the threshold value.

Consider a normal diffusion process starting @ $x_0 > 0$ with $x_t = 0$: $P_0(x) = \delta(x - x_0)$.

In Laplace space the diffusion equation is

$$u P(x,u) - \delta(x-x_0) = K \frac{\partial^2}{\partial x^2} P(x,u)$$

$$\Rightarrow P(x,u) = c_1 e^{x\sqrt{u/K}} + c_2 e^{-x\sqrt{u/K}} + \frac{\theta(x-x_0)}{2\sqrt{Ku}}\left(-e^{(x-x_0)\sqrt{u/K}} + e^{-(x-x_0)\sqrt{u/K}}\right)$$

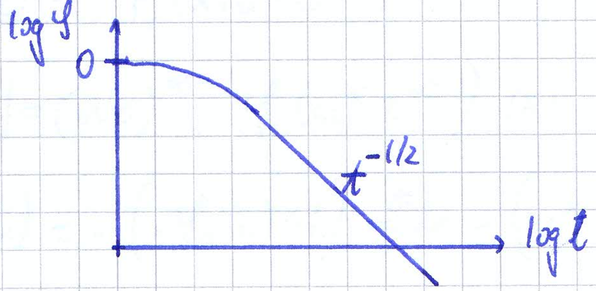
$$P(x=0,u) = 0 \Rightarrow c_1 = -c_2$$

$$\lim_{x\to\infty} P(x,u) = 0 \rightsquigarrow c_1 e^{x\sqrt{u/K}} - \frac{1}{2\sqrt{Ku}} e^{x\sqrt{u/K} - x_0\sqrt{u/K}} \overset{!}{=} 0$$

$$\Rightarrow c_1 = \frac{1}{2\sqrt{Ku}} e^{-x_0\sqrt{u/K}}$$

The survival probability becomes

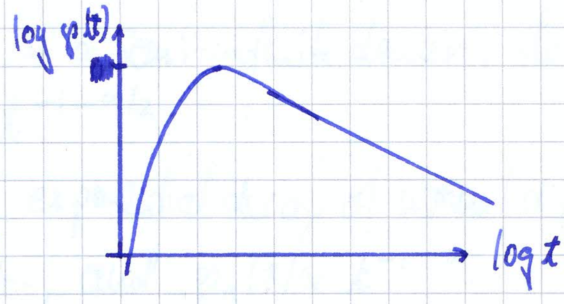
$$\mathcal{S}(t) = \int_0^\infty P(x,u)\,dx = 1 - \mathrm{erfc}\left(\frac{x_0}{\sqrt{Kt}}\right) \sim \frac{x_0}{\sqrt{\pi K t}}$$



And we obtain the first passage PDF

$$\wp(t) = -\frac{d}{dt}\mathcal{S}(t) = \frac{x_0}{\sqrt{4\pi K t^3}} e^{-\frac{x_0^2}{4Kt}} \sim \frac{x_0}{\sqrt{4\pi K t^3}} \quad \text{Lévy-Smirnov PDF}$$

$\langle t \rangle = \int_0^\infty t\,\wp(t)\,dt = \infty$! diverging mean first passage time



The same result would immediately follow from the image solution $Q(x,t) = G(x-x_0,t) - G(x+x_0,t)$ where $G(x,t)$ is the solution of the diffusion equation for initial condition $G(x,0) = \delta(x)$ (Green's function)

For random walk processes with local jump length distribution the first passage is identical to the first arrival to a point, which we describe by a sink term:

$$\frac{\partial}{\partial t}P(x,t) = K\frac{\partial^2}{\partial x^2}P(x,t) - \wp_{fa}(t)\,\delta(x)$$

such that $\wp_{fa}(t) = \wp(t)$. The first arrival PDF $\wp_{fa}$ fulfils the chain rule

$$P(-x_0,t) = \int_0^t \wp_{fa}(t')\,P(0,t-t')\,dt' \Rightarrow P(-x_0,u) = \wp_{fa}(u)\,P(0,u)$$

This relation is useful when calculating $\wp(t)$.

For <u>CTRW subdiffusion</u> we obtain a subordination relation for the first passage PDF $\wp_\alpha(t)$:

We had $P_\alpha(x,u) = (u\tau)^{\alpha-1}P_1(x,u^\alpha\tau^{\alpha-1})$

From integration: $\mathcal{S}_\alpha(u) = (u\tau)^{\alpha-1}\mathcal{S}_1(u^\alpha\tau^{\alpha-1})$ (*)

In Laplace space: $\wp_\alpha(u) = -\left(u\mathcal{S}_\alpha(u) - 1\right) \overset{(*)}{=} -\left(u^\alpha\tau^{\alpha-1}\mathcal{S}_1(u^\alpha\tau^{\alpha-1}) - 1\right)$

With $\wp_1(u) = -\left(u\mathcal{S}_1(u) - 1\right)$ we see that

$$\underline{\underline{\wp_\alpha(u) = \wp_1(u^\alpha\tau^{\alpha-1})}}$$

$\wp_\alpha(t) = \int_0^\infty \tilde{T}_\alpha(s,t)\,\wp_1(s)\,ds$:. $T_\alpha(s,u) = \exp(s u^\alpha\tau^{\alpha-1})$ characteristic fct. of a one-sided Lévy stable density