9. α-stable probability laws (continued)

9.9 Laplace transformation of one-sided PDFs

r.v. $T \geq 0$

a) Laplace tranform pair

$$p_T(t) \doteqdot \tilde{p}_T(s) \equiv \langle e^{-sT} \rangle \equiv \mathbb{E}[e^{-sT}] = \int_0^\infty e^{-st} p_T(t)\,dt$$

$\operatorname{Re} s > 0$

Normalization $\int_0^\infty p_T(t)\,dt = 1 \Rightarrow \tilde{p}_T(s=0) = 1$

Example. Exponential distribution: Most popular PDF in time domain

$$p_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0, \quad \lambda > 0, \quad \lambda \equiv 1/\tau, \Rightarrow \frac{1}{\tau} e^{-t/\tau} \\ 0, & t < 0 \end{cases}$$

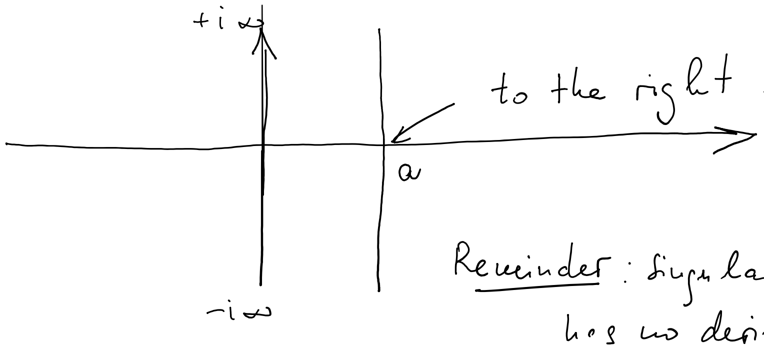
$$\lambda \int_0^\infty e^{-\lambda t}\,dt = 1, \quad \boxed{\tilde{p}_T(s) = \frac{\lambda}{\lambda + s}}$$

Property. $\tilde{p}_T(s) = \int_0^\infty e^{-st} p_T(t)\,dt \leq \int_0^\infty p_T(t)\,dt = 1$

Remind: $|\hat{p}(k)| \leq 1$

b) Inverse Laplace transform

$$p_T(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st}\, \tilde{p}_T(s)\,ds \qquad (*)$$

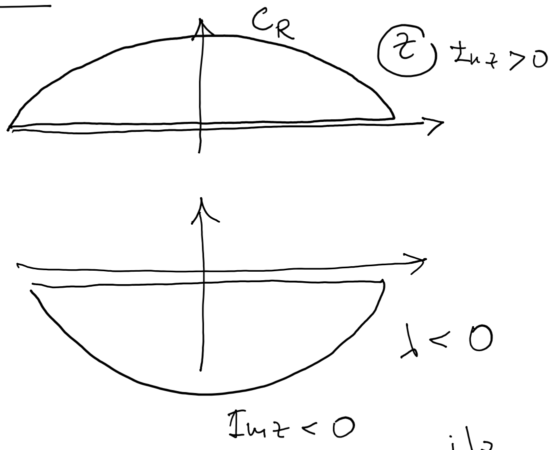


to the right from all the singular points of $\tilde{p}_T(s)$

Reminder: singular point — given function has no derivative

Exponential distribution: simple pole at $s = -\lambda$

Digression. Jordan's lemma



$$\int_{C_R} e^{i\lambda z} f(z)\, dz \to 0$$

$\lambda > 0$

if $f(z) \to 0$, as $|z| \to \infty$

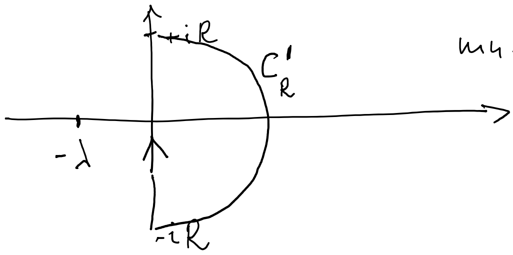
Mnemonic rule should be

$$i\lambda(i\,\mathrm{Im}\, z) = -\lambda\, \mathrm{Im}\, z < 0$$

$$e^{i\lambda z} = e^{i\lambda(i\,\mathrm{Im}\, z)} = e^{-\lambda\, \mathrm{Im}\, z}$$

$t < 0$

Go back to



mnemonic rule $e^{t\,\mathrm{Re}\, s}$

$\Downarrow$

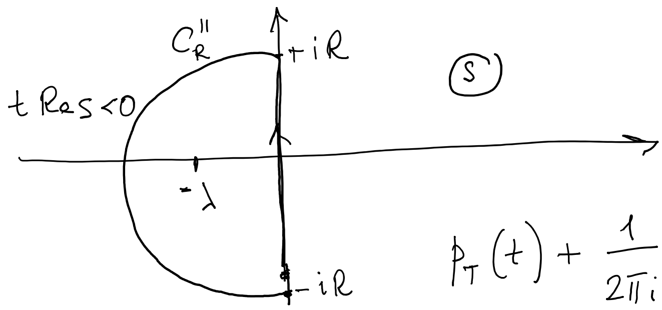
$t\,\mathrm{Re}\, s < 0$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,\frac{\lambda}{\lambda + s}\, ds + \underbrace{\int_{C'_R} \dots}_{\to 0} = 0$$

modification of Jordan's lemma

$\Rightarrow p_T(t) = 0$, $t < 0$

$t > 0$



$$p_T(t) + \frac{1}{2\pi i}\underbrace{\int_{C''_R} e^{st}\,\frac{\lambda}{\lambda+s}\, ds}_{\to 0} =$$

$$= 2\pi i\, \mathrm{Res}\{s = -\lambda\}$$

$$p_T(t) = \lambda e^{-\lambda t}, \quad t > 0$$

c) completely monotone (monotonic) functions

Def. $f(s)$ defined on $[0,\infty)$, infinitely differentiable $(0,\infty)$

$$f(s) \text{ c.m. if } (-1)^n \frac{d^n f(s)}{ds^n} \geq 0, \quad n = 0, 1, 2, \ldots$$

Examples $f(s) = e^{-s}$, $f(s) = \frac{\lambda}{\lambda + s}$, $\lambda > 0$

$f(s) = \frac{1}{s}$, $f^{(n)}(0)$ finite or infinite

Criterion of c.m. If

$$\left.\begin{array}{l} \to f(s) \text{ c.m} \\ \to g(s) > 0 \\ \to g'(s) \text{ c.m.} \end{array}\right\} \Rightarrow f(g(s)) \text{ c.m.}$$

Example. $e^{-g(s)}$ c.m.

Bernstein theorem. Function $\tilde{p}(s)$, $s \in [0,\infty)$ is the Laplace transform of the PDF $p(t)$ iff (if and only if) it is c.m. and $\tilde{p}(0) = 1$ (normalization).

Examples. 1) $\tilde{p}(s) = \frac{\lambda}{\lambda + s} \div p(t) = \lambda e^{-\lambda t}$

2) $\tilde{p}(s) = e^{-bs}$, $b > 0 \div p(t) = \delta(t - b)$

(9.10) One-sided (extreme) α-stable probability laws

$T_1, \ldots, T_n$ iid, $T_j \overset{d}{=} T$, $T, T_j \geq 0$

$$\boxed{S_n = \sum_{j=1}^{n} T_j \overset{d}{=} a_n + b_n T}, \quad (1) \quad a_n = 0, \; b_n = n^{1/\alpha}, \quad 0 < \alpha \leq 2$$

Similar to the case of symmetric stable laws:

$$\text{lhs}(1): \langle e^{sS_n} \rangle = \langle e^{s(T_1 + \ldots + T_n)} \rangle = \prod_{j=1}^{n} \langle e^{sT_j} \rangle =$$

$$= \langle e^{sT} \rangle^n = [\tilde{p}_T(s)]^n$$

$$= \langle e^{sT} \rangle^n = [\tilde{p}_T(s)]^n$$

rhs (1) $\langle e^{s n^{1/\alpha} T} \rangle = \tilde{p}_T(n^{1/\alpha} s)$

$$\boxed{[\tilde{p}_T(s)]^n = \tilde{p}_T(n^{1/\alpha} s)}$$

Suggestion. $\tilde{p}_T(s) = e^{-bs^\alpha}$, $b > 0$

Normalization $\tilde{p}_T(0) = 1$ OK

$\tilde{p}_T(s \to \infty) \to 0$ OK

Bernstein theorem. $\tilde{p}_T(s)$ must be c.m. !

Criterion. $e^{-g(s)}$ c.m. if $g(s) > 0$ and $g'(s)$ c.m.

Check. $g(s) = bs^\alpha > 0$ OK $[0, \infty)$

$\varphi(s) = g'(s) = \alpha s^{\alpha - 1}$ c.m. ?

Must be $(-1)^n \dfrac{d^n \varphi(s)}{ds^n} \geqslant 0$ $\quad \varphi'(s) = g''(s) =$

$= \alpha(\alpha - 1) s^{\alpha - 2}$

$\varphi'(s) < 0$, $0 < \alpha < 1$ c.m.

$\varphi''(s) > 0$ $\quad 1 < \alpha < 2$ not c.m.

Conclusion. $\boxed{\tilde{p}_T(s) = e^{-bs^\alpha}, \quad 0 < \alpha < 1}$

b) explicit form of the Fourier transform

$$p(t) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} e^{st - bs^\alpha} ds$$

Branch point $s = 0$, but the integrand is bounded $\Rightarrow$
$\Rightarrow$ make a shift to $a = 0$, since no singular points are crossed; the contour only touches the branch point

$$p(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st - bs^\alpha} ds = \{ s = -ik \} =$$

$$= \frac{1}{2\pi i} \int_{+\infty}^{-\infty} e^{-ikt - b(-ik)^\alpha} d(-ik) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikt - b(-ik)^\alpha} dk \Rightarrow$$

$$2\pi i \quad {}_{+\infty} \qquad\qquad 2\pi \ {}_{-\infty}$$

$$= \Big\{ b(-ik)^\alpha = b\left[|k|\,\mathrm{sgn}(k)\, e^{-\frac{i\pi}{2}}\right]^\alpha =$$

$$= b|k|^\alpha \left[\mathrm{sgn}(k)\left(\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} - i\sin\left(\tfrac{\pi}{2}\right)\right)\right]^\alpha =$$

$$= b|k|^\alpha \left[-i\sin\left(\frac{\pi}{2}\mathrm{sgn}(k)\right)\right]^\alpha = b|k|\left[e^{-i\frac{\pi}{2}\mathrm{sgn}\,k}\right]^\alpha =$$

$$= \underbrace{b|k|^\alpha e^{-\frac{i\pi\alpha}{2}\mathrm{sgn}(k)}} \Big\} \Longrightarrow$$

$$\Longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikt - b|k|^\alpha \exp\left[-i\frac{\pi\alpha}{2}\mathrm{sgn}(k)\right]}\, dk$$

Notation $l_{\alpha,1}(t)$ α-stable one-sided

$$\tilde{l}_{\alpha,1}(s) = e^{-bs^\alpha} \longleftrightarrow \hat{l}_{\alpha,1}(k) = e^{-b|k|^\alpha e^{-(i\pi\alpha/2)\,\mathrm{sgn}(k)}} =$$

$$= e^{-b|k|^\alpha\left[\cos\frac{\pi\alpha}{2} - i\,\mathrm{sgn}(k)\sin\frac{\pi\alpha}{2}\right]} =$$

$$= \exp\Big[\underbrace{-b\cos\left(\frac{\pi\alpha}{2}\right)}_{\sigma^\alpha}|k|^\alpha\left(1 - i\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\Big]$$

$$\boxed{\hat{l}_{\alpha,1}(k) = \exp\left[-\sigma^\alpha|k|^\alpha\left(1 - i\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right]} \quad (**)$$

Compare symmetric α-stable

$$\hat{l}_\alpha(k) = e^{-\sigma^\alpha|k|^\alpha}$$

Reminder. General form of the CF of α-stable probability law

$$\hat{l}_{\alpha,\beta}(k;\sigma,\mu) = \exp\left[i\mu k - \sigma^\alpha|k|^\alpha\left(1 - i\beta\,\mathrm{sgn}(k)\,\omega(k,\alpha)\right)\right]$$

$$-1 \le \beta \le 1, \quad 0 < \alpha \le 2, \quad \sigma > 0, \quad \mu \text{ is } \mathrm{Re}$$

$$\omega(k,\alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \alpha \ne 1, \quad 0 < \alpha \lesssim 2 \\ -\frac{2}{\pi}\ln|k|, & \alpha = 1 \end{cases}$$

<u>Remark 1</u>. $(**)$ corresponds to $\beta = 1$

Remark 1. $(**)$ corresponds to $\beta = 1$

Remark 2. Similarly to symmetric α-stable laws

$$\ell_{\alpha,\beta}(x;\sigma,\mu) = \frac{1}{\sigma}\,\ell_{\alpha,\beta}\left(\frac{x-\mu}{\sigma};1,0\right)$$

Notation $\ell_{\alpha,\beta}(x;1,0) = \ell_{\alpha,\beta}(x)$

Remark 3. Important symmetry property

$$\boxed{\ell_{\alpha,-\beta}(x) = \ell_{\alpha,\beta}(-x)} \qquad (***)$$

For instance,

→ Asymptotics of $\ell_{\alpha<1,1}(x)$, $x \to 0^+$ $(x>0)$ =
= Asymptotics of $\ell_{\alpha<1,-1}(x)$, $x \to 0^-$ $(x<0)$

→ The same for $\ell_{\alpha<1,1}(x)$, $x \to \infty$ AND
$\ell_{\alpha<1,-1}(x)$, $x \to -\infty$

Proof of $(***)$: Assignment 2

- Support of PDFs $\quad \ell_{\alpha<1,1}(x) \neq 0$ at $0 \le x < \infty$
$\ell_{\alpha<1,-1}(-x) \neq 0$ at $-\infty < x \le 0$

Proof. $\tilde{\ell}_{\alpha,1}(s) = e^{-bs^\alpha}$, $s=0$ singular point (branch point)
similar to $\tilde{\ell}(s) = \frac{\lambda}{\lambda+s}$ (see above)

9.11 Asymptotics of one-sided α-stable laws at $t \to \infty$

$$\tilde{\ell}_{\alpha,1}(s) = e^{-bs^\alpha} \simeq 1 - bs^\alpha, \quad s \to 0, \quad 0<\alpha<1$$

Reminder. For symmetric laws

$$\hat{\ell}_\alpha(k) = e^{-\sigma^\alpha |k|^\alpha} \simeq 1 - \sigma^\alpha |k|^\alpha, \quad k \to 0, \quad 0<\alpha<2$$

$$\Updownarrow$$

$$\ell_\alpha(x) \sim \frac{C_1(\alpha)\,\sigma^\alpha}{|x|^{1+\alpha}}, \quad C_1(\alpha) = \frac{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)}{\pi}$$

$$l_\alpha(x) \sim \frac{C_1(\alpha)\,b}{|x|^{1+\alpha}}, \quad C_1(\alpha) = \frac{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)}{\pi}$$

By analogy $\underline{\underline{l_{\alpha,1}(t) \sim \frac{A(\alpha)\tau^\alpha}{t^{1+\alpha}}}}$, τ^α comes from normalization

$$\int_0^\infty l_{\alpha,1}(t)\,dt = 1$$

$$\Downarrow$$

$$[l_{\alpha,1}] = \frac{1}{\text{time}}$$

! Tauberian theorem for Laplace transform

Let $L(t)$ slowly varying function

$$\lim_{t\to\infty} \frac{L(ct)}{L(t)} = 1 \quad \text{for any } c>0$$

Example $L(t) = \ln t$

$$\boxed{f(t) \underset{t\to\infty}{\sim} t^{\rho-1}L(t), \quad 0<\rho<\infty \iff \tilde{f}(s) \underset{s\to 0}{\sim} \Gamma(\rho)s^{-\rho}L(1/s)}$$

Diret application to $l_{\alpha,1}(t)$ is not possible: $l_{\alpha,1}(t) \sim \frac{\tau^\alpha}{t^{1+\alpha}}$.

$L(t)=1$, $\rho-1 = -1-\alpha$, $\Rightarrow \rho = -\alpha \Rightarrow \tilde{l}_{\alpha,1}(s) \sim s^\alpha$

BUT $\tilde{l}_{\alpha,1}(0)$ must be 1 !

Digression Cumulative distr. function

$$P(t) = \int_0^t p(t')\,dt' \implies \tilde{P}(s) = \frac{\tilde{p}(s)}{s}$$

Survival probability $\Psi(t) = 1 - P(t) = \int_t^\infty p(t')\,dt'$

$$\tilde{\Psi}(s) = \frac{1}{s} - \frac{\tilde{p}(s)}{s} = \frac{1-\tilde{p}(s)}{s}$$

End of digression

- Go back to α-stable: CDF $L_{\alpha,1}(t) = \int_0^t l_{\alpha,1}(t')\,dt'$

$$1 - L_{\alpha,1}(t) = \int_t^\infty l_{\alpha,1}(t')\,dt' \underset{t\to\infty}{\sim} A(\alpha)\tau^\alpha \int_t^\infty (t')^{-1-\alpha}\,dt' =$$

$$= \frac{A(\alpha)\tau^\alpha}{\alpha t^\alpha}, \quad t\to\infty$$

$$= \frac{A(\alpha)\tau^\alpha}{\alpha t^\alpha}, \quad t \to \infty$$

- Apply Tauberian theorem to $\Psi_{\alpha,1}(t) = 1 - L_{\alpha,1}(t)$:

$\alpha = 1-\rho \Rightarrow \rho = 1-\alpha, \quad 0<\rho<1$

$$\tilde{\Psi}_{\alpha,1}(s) \underset{\substack{\uparrow \\ \text{Tauberian}}}{\sim} \Gamma(1-\alpha) s^{\alpha-1} \frac{A(\alpha)\tau^\alpha}{\alpha}, \quad s \to 0$$

- Employ $\tilde{\Psi}_{\alpha,1}(s) = \frac{1-\tilde{\ell}_{\alpha,1}(s)}{s}$

$$1 - \tilde{\ell}_{\alpha,1}(s) = s\tilde{\Psi}_{\alpha,1}(s) \underset{s\to 0}{\sim} \Gamma(1-\alpha)\frac{A(\alpha)\tau^\alpha}{\alpha} s^\alpha \Rightarrow$$

$$\Rightarrow \tilde{\ell}_{\alpha,1}(s) \sim 1 - bs^\alpha, \quad s \to 0$$

$$b = \frac{A(\alpha)\Gamma(1-\alpha)}{\alpha}\tau^\alpha$$

Conclusion

$$\boxed{\tilde{\ell}_{\alpha,1}(s) = e^{-bs^\alpha} \underset{s\to 0}{\sim} 1 - bs^\alpha \Longleftrightarrow \ell_{\alpha,1}(t) \underset{t\to\infty}{\sim} \frac{A(\alpha)\tau^\alpha}{t^{1+\alpha}}, \quad 0<\alpha<1}$$

<u>Remark</u> Similarity of asymptotics for sym. and extr. distr.

Remember $b\cos\left(\frac{\pi\alpha}{2}\right) = \sigma^\alpha$

$$b\cos\left(\frac{\pi\alpha}{2}\right) = \frac{A(\alpha)\Gamma(1-\alpha)}{\alpha}\tau^\alpha \cos\left(\frac{\pi\alpha}{2}\right) = \sigma^\alpha$$

$$\ell_{\alpha,1}(t) \sim \frac{A(\alpha)\tau^\alpha}{t^{1+\alpha}} = \frac{\alpha\sigma^\alpha}{\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(1-\alpha)\, t^{1+\alpha}} =$$

$$= \left\{ \Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin(\pi\alpha)} \right\} = \frac{\alpha\sigma^\alpha}{\cos\left(\frac{\pi\alpha}{2}\right)} \frac{\Gamma(\alpha)\sin(\pi\alpha)}{\pi t^{1+\alpha}} =$$

$$= \left\{ \sin(\pi\alpha) = 2\cos\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right), \quad \Gamma(1+\alpha) = \alpha\Gamma(\alpha) \right\} \Rightarrow$$

$$\Rightarrow \ell_{\alpha,1}(t) \sim \frac{2C_1(\alpha)\sigma^\alpha}{t^{1+\alpha}}$$

Conclusion

$$\hat{l}_{\alpha,1}(k) = \exp\left[-\sigma^\alpha |k|^\alpha \left(1 - i\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right] \to$$

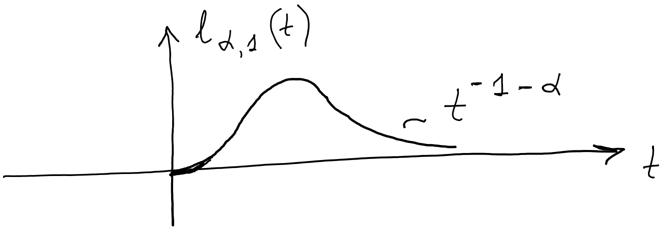
$$\to l_{\alpha,1} \sim \frac{2C_1(\alpha)\sigma^\alpha}{t^{1+\alpha}}, \quad 0<\alpha<1$$

$$C_1(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)$$ The same as for symmetric PDFs

$$\left(l_\alpha(x;\sigma,0) \sim C_1(\alpha)\frac{\sigma^\alpha}{|x|^{1+\alpha}}, \quad |x|\to\infty \right)$$

- Asymptotics at $t\to 0$ $\quad l_{\alpha,1}(t) \sim \exp\left(-\frac{C_2(\alpha)}{t^{\alpha/(1-\alpha)}}\right)$

$$l_{\alpha,1}^{(n)}(t)\Big|_{t=0} = 0, \quad n = 0, 1, \ldots$$



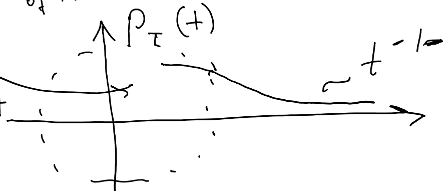
- GCLT for one-sided laws

iid $T_1, \ldots, T_n \geqslant 0 \quad p_T(t) \sim \frac{A(\alpha)\tau^\alpha}{t^{1+\alpha}}, \; t\to\infty$

$$Y = \frac{1}{n^{1/\alpha}}\sum_{j=1}^{n} T_j$$

$$p_Y(t) \xrightarrow[n\to\infty]{} l_{\alpha,1}(t) \sim \frac{A(\alpha)\tau^\alpha}{t^{1+\alpha}}, \quad t\to\infty$$

The shape of central part of PDF is not important



$\Longrightarrow$

Remark. 1-st moment $\int_0^\infty t\, \ell_{\alpha,1}(t)\,dt = \infty$, $0<\alpha<1$

q-th moment $= \langle t^q \rangle = \begin{cases} C(q,\alpha), & q<\alpha \\ \infty, & q\geq\alpha \end{cases}$