Lectures 5 - 6

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8. Central Limit Theorem (CLT)

A pedestrian approach.

Universal features of sums of i.r.v.

(8.1) Gaussian iid r.v.

$X_1, \dots, X_n$ iid r.v. $\quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}, \quad Var\{x\} = \sigma^2$

$\langle X \rangle = 0$

$$\hat{p}(k) = \langle e^{ikX} \rangle = e^{-\sigma^2 k^2/2}$$

(a) $Y = X_1 + \dots + X_n$, $\quad p_Y(y) = ?$

$$\hat{p}_Y(k) = \langle e^{ikY} \rangle = \langle e^{ik(X_1 + \dots + X_n)} \rangle = \langle e^{ikX} \rangle^n =$$

$$= [\hat{p}(k)]^n = e^{-n\sigma^2 k^2/2} \div p_Y(y) = \frac{1}{\sqrt{2\pi n\sigma^2}} \exp\left(-\frac{y^2}{2n\sigma^2}\right) \Rightarrow$$

$$\Rightarrow \sigma_Y^2 = n\sigma^2 \xrightarrow[n\to\infty]{} \infty$$

(b) Arithmetic mean $Y_A = \frac{Y}{n} = \frac{X_1 + \dots + X_n}{n}$.

$$\hat{p}_{Y_A}(k) = \langle e^{ikY_A} \rangle = \langle e^{ik\frac{Y}{n}} \rangle = \langle e^{i\frac{k}{n}Y} \rangle =$$

$$= \langle e^{i\frac{k}{n}X} \rangle^n = e^{-\frac{1}{2}\frac{\sigma^2}{n}k^2} \div p_{Y_A}(y) = \sqrt{\frac{1}{2\pi\sigma^2/n}} \exp\left(-\frac{ny^2}{2\sigma^2}\right)$$

$$\Rightarrow \sigma_{Y_A}^2 = \frac{\sigma^2}{n} \xrightarrow[n\to\infty]{} 0$$

(c) suitably scaled sum $\quad Z = \frac{X_1 + \dots + X_n}{\sqrt{n}} = \frac{Y}{\sqrt{n}}$

$$\hat{p}_Z(k) = \langle e^{ikZ} \rangle = \langle e^{ik\frac{Y}{\sqrt{n}}} \rangle = \langle e^{i\frac{k}{\sqrt{n}}Y} \rangle = \langle e^{i\frac{k}{\sqrt{n}}X} \rangle^n =$$

$$= \left(e^{-\frac{\sigma^2 k^2}{2n}}\right)^n = e^{-\sigma^2 k^2/2} = \hat{p}_X(k)$$

Conclusion. $X_1, \dots, X_n$ iid Gaussian r.v.

$\langle X \rangle = 0$, $Var[X] = \sigma^2$, then

$$Z = \frac{X_1 + \dots + X_n}{\sqrt{n}} \Rightarrow p_Z(x) \text{ Gaussian}, \ \langle Z \rangle = 0, \ Var[Z] = \sigma^2$$

(8.2) $X_1, \dots, X_n$ iid r.v. <u>non-Gaussian</u>

$$\langle X \rangle = 0 \quad , \quad Var[X] = \sigma^2$$

$$Z = \frac{X_1 + \dots + X_n}{\sqrt{n}} \quad , \quad p_Z(z) \underset{n\to\infty}{=} ?$$

$$\hat{p}_X(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\, dx = \int_{-\infty}^{\infty} \Big(\underset{=}{1} + \underset{\sim}{ikx} + \underset{\approx}{\frac{1}{2!}(ikx)^2} + \frac{(ikx)^3}{3} + \dots \Big) p(x)\, dx$$

$$= 1 - \frac{\sigma^2 k^2}{2} + O(k^3) \quad \text{(main contribution from small } k\text{!)}$$

$$\hat{p}_Z(k) = \langle e^{\frac{ik}{\sqrt{n}}(X_1 + \dots + X_n)} \rangle = \langle e^{i\frac{k}{\sqrt{n}}X} \rangle^n =$$

$$= \left(1 - \frac{\sigma^2 k^2}{2n} + O\left(\frac{k^3}{n^{3/2}}\right)\right)^n \underset{\uparrow}{\overset{n\to\infty}{=}} e^{-\frac{\sigma^2 k^2}{2}} \Rightarrow$$

$$\lim_{n\to\infty} \left(1 \pm \frac{\alpha}{n}\right)^n = e^{\pm\alpha}$$

$$\Rightarrow \quad p_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right)$$

<u>Central Limit Theorem</u>. (Lévy-Lindeberg)

If $X_1, \dots, X_n$ iid r.v. $p_X(x)$, $\langle X \rangle = 0$, $Var[X] = \sigma^2$

then, properly normalized sum $Z = \frac{1}{\sqrt{n}}(X_1 + \dots + X_n)$

$$p_Z(z) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right), \quad \text{same } \sigma^2.$$

<u>Remark 1</u>. Two important conditions: 1) the summands X_i are iid
2) the variance is finite

<u>Remark 2</u>. No smoothness of $p(x)$ are required.

Example. Discrete-time random walk

$$p(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$$

<u>Remark 3</u>. Alternative form $\langle X \rangle = \mu \neq 0$

$$Z_1 = \frac{1}{\sigma\sqrt{n}}\left[\sum_{j=1}^{n} X_j - n\mu\right] \Rightarrow p_{Z_1}(z) \Longrightarrow \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

standard normal PDF

properly normalized (scaled) AND centered

OR (integral form) $\quad \Pr\{Z_1 \le x\} \longrightarrow \int_{-\infty}^{x} g(x)\,dx$

OR (integral form) $\Pr\{Z_1 \leq x\} \longrightarrow \int_{-\infty} g(x)dx$

(8.3) $X_1, \dots X_n$ independent, but not identically distributed

(Lindeberg – Feller form of CLT)

$X_1, \dots, X_n$ irv $\langle X_i \rangle = 0$, $Var[X_i] = \sigma_i^2$

Take $B_n^2 = \sum_{j=1}^{n} \sigma_j^2 = Var\left[\sum_{j=1}^{n} X_j\right]$

Compose $Y = \frac{1}{B_n}\sum_{i=1}^{n} X_j$

If for $\forall \varepsilon > 0$ $\lim_{n\to\infty} \frac{1}{B_n^2}\sum_{i=1}^{n} \int_{|x|>\varepsilon B_n} x^2 p(x) = 0$ necessary and sufficient

$\Rightarrow$ $p_Y(y) \xrightarrow[n\to\infty]{} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$

Example. Consider Laplace $\sim e^{-|x|}$, or $\sim 1/x^4$, $x \to \infty$

Meaning. The "wild" values of r.v. compared with B_n are insignificant and can be truncated without affecting the general behavior of the sums.

Physical consequence CLT: Gaussian probability naturally appears if evolution of the system or result of experiment are determined by the sum of random factors.

(8.4) $Var[X]$ is infinite ?

Example. Cauchy distribution

$X_1, \dots, X_n$ iid r.v. $p_X(x) = \frac{b}{\pi(x^2+b^2)} \div \hat{p}_X(k) = e^{-b|k|}$

(a) sum $Y = X_1 + \dots + X_n$

$\hat{p}_Y(k) = \langle e^{-b|k|} \rangle^n = e^{-bn|k|} \div p_Y(k) = \frac{bn}{\pi(x^2+b^2n^2)}$

Cauchy with $b \to bn$

(b) as in CLT $Z = \frac{Y}{\sqrt{n}}$

$\hat{p}_Z(k) = \langle e^{i\frac{k}{\sqrt{n}}Y} \rangle = \langle e^{i\frac{k}{\sqrt{n}}X} \rangle^n = e^{-b\sqrt{n}|k|} \div$

$\div\, p_Z(z) = \frac{b\sqrt{n}}{\pi(x^2+b^2n)}$: Cauchy at any n with $b \to b\sqrt{n}$

no Gaussian at $n \to \infty$

~~$\div\ p_Z(z) = \frac{b\sqrt{n}}{\pi(x^2+b^2 n)}$: Cauchy at any n with $b \to b\sqrt{n}$, no Gaussian at $n\to\infty$~~

BUT:

(c) properly normalized sum $Y_A = \frac{Y}{n}$ (arithmetic mean)

$$\hat{p}_{Y_A}(k) = \langle e^{i\frac{k}{n}Y}\rangle = \langle e^{i\frac{k}{n}X}\rangle^n = \left(e^{-\frac{b}{n}|k|}\right)^n = e^{-b|k|}$$

Cauchy $b \to b$

Gaussian X_i, $Var[X_i] = \sigma^2$	Cauchy X_i, characterised by $b>0$
$Y = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$	$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$
$p_Y(y) = $ Gaussian, $Var[Y] = \sigma^2$	$p_Y(y) = $ Cauchy, same b

CLT $\Longrightarrow$ Generalised CLT

9. Stable (α-stable) probability laws and Generalised CLT

9.1 Definition of stable r.v.

Let $X, X_1, \ldots, X_n$ iid, $X_j \overset{d}{=} X$

X stable if $\exists\, b_n > 0,\ a_n$

$$\boxed{S_n = \sum_{j=1}^{n} X_j \overset{d}{=} a_n + b_n X} \qquad (1)$$

X strictly stable if $a_n = 0$

Example 1 Gaussian $p_X(x) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{x^2}{4\sigma^2}} \div \hat{p}_X(k) = e^{-\sigma^2 k^2}$

Note: $Var\{X\} = 2\sigma^2$

$$lhs(1) \Rightarrow \langle e^{ikS_n}\rangle = \langle e^{ikX}\rangle^n = e^{-n\sigma^2 k^2}$$

$$rhs(1) \Rightarrow \langle e^{ik(a_n+b_nX)}\rangle = e^{ika_n}\langle e^{ikb_nX}\rangle = e^{ika_n}\hat{p}_X(b_n k) = e^{ika_n - b_n^2\sigma^2 k^2}$$

$\Rightarrow a_n = 0,\ b_n = \sqrt{n} \Rightarrow$ strictly stable

Example 2. Cauchy $p_X(x) = \frac{b}{\pi(x^2+b^2)} \div \hat{p}_X(k) = e^{-b|k|}$

$lhs(1)\ \langle e^{ikS_n}\rangle \ldots e^{-nb|k|}$

$\ldots \; p_X(x) = \ldots \frac{\;}{\pi(x^2+b^2)} \cdot \; p_X \ldots$

$$\left.\begin{aligned} &\text{lhs}(1) \Rightarrow \langle e^{ikS_n}\rangle = e^{-nb|k|} \\ &\text{rhs}(1) \rightarrow e^{ika_n - bb_n|k|}. \end{aligned}\right\} \Rightarrow \begin{aligned} &a_n = 0,\; b_n = n \\ &\text{strictly stable} \end{aligned}$$

Example 3. Laplace $\quad p_X(x) = \frac{1}{2b} e^{-\frac{|x|}{b}} \;\div\; \hat{p}_X(k) = \frac{1}{1+b^2k^2}$

$$\text{lhs}(1) \Rightarrow \langle e^{ikS_n}\rangle = \frac{1}{(1+b^2k^2)^n}$$

$$\text{rhs}(1) \Rightarrow \langle e^{ik(a_n + b_n X)}\rangle = e^{ika_n}\hat{p}_X(b_n k) = \frac{e^{ika_n}}{1+b_n^2 b^2 k^2}$$

non-stable

Theorem (Feller vol. II) $\quad \boxed{b_n = n^{1/\alpha}}$ $\quad \alpha$ index of stability, Lévy index

Gaussian $\alpha = 2$

Cauchy $\alpha = 1$

9.2 Symmetric stable laws. Characteristic function.

$$l_\alpha(x) = l_\alpha(-x)$$

strictly stable: symmetric, $a_n = 0$

CF: $\hat{p}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\,dx, \quad p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{p}(k)\,dk$

$$\hat{p}^*(k) = \int_{-\infty}^{\infty} e^{-ikx} p(x)\,dx = \hat{p}(-k) \quad \text{general property of CF}$$

Since $p(x) = p(-x) \Longrightarrow \underline{\hat{p}(-k)} = \int_{-\infty}^{\infty} e^{-ikx} p(-x)\,dx = \left\{\begin{aligned} x' &= -x \\ dx' &= -dx' \end{aligned}\right\} =$

$$= \int_{-\infty}^{\infty} e^{ikx'} p(x')\,dx' = \underline{\hat{p}(k)} \quad \text{symmetric CF}$$

From Eq (1), $b_n = n^{1/\alpha}$, $a_n = 0$.

$$\langle e^{ik(X_1 + \ldots + X_n)}\rangle = \langle e^{ikn^{1/\alpha}X}\rangle$$

$$[\hat{l}_\alpha(k)]^n = \hat{l}_\alpha(kn^{1/\alpha})$$

Guess $\hat{l}_\alpha(k) = e^{-ak^\alpha}$, $a > 0, \alpha > 0$? $\quad \hat{p}^*(k) \neq \hat{p}(-k) \quad$ NO

Take $\quad \boxed{\hat{l}_\alpha(k) = e^{-\sigma^\alpha |k|^\alpha}} \;,\; \sigma > 0 \qquad (2)$

~~Take~~ $\cancel{\hat{\ell}_\alpha(k) = e^{-\sigma^\alpha |k|^\alpha}}$ ~~, $\sigma > 0$~~

a) normalization $\hat{\ell}_\alpha(0) = 1 \Rightarrow \alpha > 0$

b) 1 st moment $\langle X \rangle = \frac{1}{i} \frac{d\hat{\ell}_\alpha(k)}{dk}\Big|_{k=0}$

2 nd moment $\langle X^2 \rangle = -\frac{d^2\hat{\ell}_\alpha(k)}{dk^2}\Big|_{k=0}$

Take 1 st derivative $\frac{d}{dk} e^{-\sigma^\alpha |k|^\alpha} \underset{\uparrow}{=} -\alpha\sigma^\alpha |k|^{\alpha-1} e^{-\sigma^\alpha|k|^\alpha} \operatorname{sgn} k$

$\frac{d|k|}{dk} = \operatorname{sgn} k$

Take 2nd derivative $\frac{d^2}{dk^2} e^{-\sigma^\alpha|k|^\alpha} = -\alpha \underbrace{(\alpha-1)\sigma^\alpha |k|^{\alpha-2} \operatorname{sgn} k}_{\frac{d}{dk}|k|^{\alpha-1}} e^{-\sigma^\alpha|k|^\alpha} \operatorname{sgn} k -$

$-\alpha\sigma^\alpha |k|^{\alpha-1} (-\alpha\sigma^\alpha |k|^{\alpha-1} \operatorname{sgn} k) e^{-\sigma^\alpha|k|^\alpha} \operatorname{sgn} k -$

$-\alpha\sigma^\alpha |k|^{\alpha-1} e^{-\sigma^\alpha|k|^\alpha} \underbrace{2\delta(k)}_{\frac{d}{dk}\operatorname{sgn} k} \Rightarrow$

$\Rightarrow \langle X^2 \rangle = \left\{ \alpha\sigma^\alpha |k|^{\alpha-2} [\alpha - 1 - \alpha\sigma^\alpha |k|^\alpha + 2|k|\delta(k)] e^{-\sigma^\alpha|k|^\alpha} \right\}_{k=0} =$

$= 0$ if $\alpha > 2$

BUT $\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx > 0$

Conclusion $\boxed{\hat{\ell}_\alpha(k) = e^{-\sigma^\alpha|k|^\alpha}, \quad 0 < \alpha \leq 2}$

9.3 Symmetric stable laws. PDFs and moments

a) PDF via elementary functions for $\alpha = 1$ (Cauchy) and $\alpha = 2$ (Gauss)

b) PDF in general case via special functions Fox H functions

c) $\langle X^2 \rangle = \infty, \quad 0 < \alpha < 2$

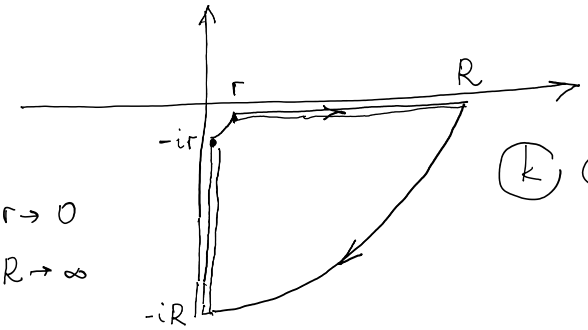
$\langle X^2 \rangle = 2\sigma^2, \quad \alpha = 2$ Gauss $\hat{\ell}_2(k) = e^{-\sigma^2 k^2}$

$\langle X \rangle = 0$

d) PDF asymptotics at $x \to \infty$, $0 < \alpha < 2$

$$l_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - \sigma^\alpha |k|^\alpha} dk = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{-ikx - \sigma^\alpha k^\alpha} dk \qquad (*)$$

Consider $x > 0$



(k) Complex plane

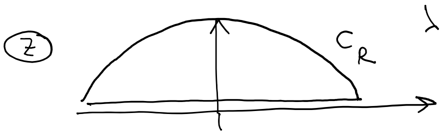
$$\boxed{\int_0^\infty + \int_{C_R} + \int_{-i\infty}^{-i0} + \int_{C_r} = 0}\,, \quad r \to 0, \; R \to \infty$$

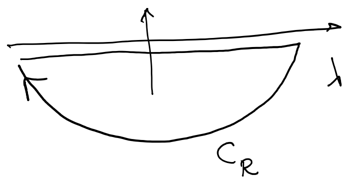
Jordan's lemma $\quad \lim_{R\to\infty} \int_{C_R} = 0$

Digression. Jordan's lemma $\quad \int_{C_R} e^{i\lambda z} f(z)\, dz \to 0$

if $f(z) \to 0$, $|z| \to \infty$

$\lambda > 0$





$\lambda < 0$

$$\int_{C_r} = \lim_{r\to 0} \int_{C_r} e^{-ikx - \sigma^\alpha k^\alpha} dk = \left\{ \begin{array}{l} k = re^{i\theta}, \; -\frac{\pi}{2} \le \theta < 0, \\ dk = ire^{i\theta} d\theta \end{array} \right\} =$$

$$= \lim_{r\to 0} ir \int_{-\pi/2}^{0} d\theta\, e^{i\theta} e^{ixre^{i\theta} - \sigma^\alpha r^\alpha e^{i\alpha\theta}} = 0$$

$$\int_0^\infty e^{-ikx - \sigma^\alpha k^\alpha} dk = -\int_{-i\infty}^{-i0} e^{-ikx - \sigma^\alpha k^\alpha} dk = \{ ik = æ \} =$$

$$= -\frac{1}{i} \int_{-\infty}^{0} e^{-xæ - \sigma^\alpha æ^\alpha \underbrace{e^{-i\pi\alpha/2}}_{-i}} dæ \longrightarrow \{ æx = y \} \to (*)$$

$$l_\alpha(x) = \frac{1}{\pi x} \operatorname{Re} \left\{ -i \int_0^\infty \exp\left[-y - \frac{\sigma^\alpha y^\alpha}{x^\alpha} e^{-i\pi\alpha/2} \right] dy \right\} =$$

$$= \frac{1}{\pi x} \operatorname{Re} \left\{ -i \int_0^\infty e^{-y} \left[1 - \frac{\sigma^\alpha y^\alpha}{x^\alpha} e^{-i\pi\alpha/2} + O\left(\frac{y^{2\alpha}}{x^{2\alpha}} \right) \right] dy \right\} \simeq$$

$$\cancel{\simeq \frac{\sigma^\alpha}{\pi} \operatorname{Re}\left\{ i e^{-i\pi\alpha/2} \right\} \int_0^\infty e^{-y} y^\alpha dy \,, \quad x \to \infty}$$

$$\simeq \frac{\sigma^\alpha}{\pi x^{1+\alpha}} \,\mathrm{Re}\left\{ i \underbrace{e^{-i\pi\alpha/2}}_{\cos - i \sin} \right\} \underbrace{\int_0^\infty e^{-y} y^\alpha \, dy}_{\Gamma(1+\alpha)}, \qquad x \to \infty$$

$$\boxed{l_\alpha(x) \approx \frac{C(\alpha)}{|x|^{1+\alpha}}, \quad C(\alpha) = \frac{\sigma^\alpha}{\pi} \sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)} \qquad 0<\alpha<2$$

Remark 1 Power-law asymptotics with strict inequality $\alpha<2$

at $\alpha=2$ $\quad \hat{l}_2(k) = e^{-\sigma^2 k^2} \Rightarrow l_2(x) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left(-\frac{x^2}{4\sigma^2}\right)$

Remark 2 Second moment $= \infty$

$$\lim_{A_1, A_2 \to 0} \int_{A_1}^{A_2} x^2 l_\alpha(x)\,dx = \int_0^{A_1} x^2 l_\alpha(x)\,dx + \int_0^{A_2} x^2 l_\alpha(x)\,dx \sim$$

$$\sim \int_0^{A_1} x^2 x^{-\alpha-1}\,dx + \int_0^{A_2} x^2 x^{-\alpha-1}\,dx \sim A_1^{2-\alpha} + A_2^{2-\alpha} \to \infty \quad A_1, A_2 \to \infty$$

Remark 3. q-th moment

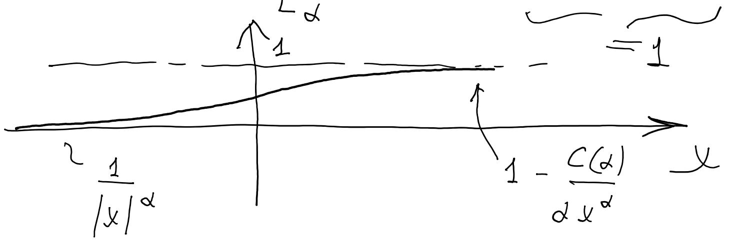
$$\langle |x|^q \rangle = \int_{-\infty}^{\infty} |x|^q l_\alpha(x)\,dx \sim \int^{A} |x|^q |x|^{-\alpha-1}\,dx \sim$$

$$\sim \int_0^A x^{q-1-\alpha}\,dx \sim \begin{cases} A^{q-\alpha} \nearrow \text{finite}, & q<\alpha \\ \qquad \searrow \text{infinite}, & q>\alpha \\ \ln A, & q=\alpha \end{cases}$$

$$\langle |x|^q \rangle = \begin{cases} C(q,\alpha), & q<\alpha \\ \infty, & q \geq \alpha \end{cases} \qquad q>0$$

Remark 4. Asymptotics of cumulative probability

$$L_\alpha(x) = \int_{-\infty}^{x} l_\alpha(y)\,dy \sim \begin{cases} \frac{C(\alpha)}{\alpha |x|^\alpha}, & x \to -\infty \\ \underbrace{\int_{-\infty}^{\infty} l_\alpha(y)\,dy}_{=1} - \int_x^\infty l_\alpha(y)\,dy \sim 1 - \frac{C(\alpha)}{\alpha x^\alpha} \end{cases}$$



Remark 5.