II. 8. Fractional Brownian motion $B_H(t)$ (continued)

Reminder. Structure function (Kolmogorov 1941)

$$S_H(\tau) = \langle (B_H(t+\tau) - B_H(t))^2 \rangle = A\tau^{2H}, \text{ no } t\text{-dep.}$$

Mean squared increments

We proved $0 < H \leq 1$

$H = 1$ $\quad B_1(t) = v t + c$ random line

$$S_1(\tau) = \langle v^2 \rangle \tau^2$$

(8.2) 1) $\langle (B_H(t_2) - B_H(t_1))^2 \rangle = |t_2 - t_1|^{2H}$

up to prefactor

Take $t_1 = 0$, $B_H(0) = 0$ (usual in theory of stoch proc)

2) $MSD = \langle B_H^2(t) \rangle = t^{2H}$

- $0 < H < 1/2$ subdiffusion
- $1/2 < H \leq 1$ superdiff
- $H = 1/2$ normal diff. oBm (Wiener)
- $H = 1$ ballistic motion

3) ACF

Take 1) $\Rightarrow \langle B_H^2(t_2) \rangle - 2\langle B_H(t_1) B_H(t_2) \rangle + \langle B_H^2(t_1) \rangle =$

$= |t_2 - t_1|^{2H}$

Use 2) $\Rightarrow$ $\boxed{\langle B_H(t_1) B_H(t_2) \rangle = \frac{1}{2}\left[t_1^{2H} + t_2^{2H} - |t_2 - t_1|^{2H}\right]}$

Check $H = 1/2$ $\quad \langle B_{1/2}(t_1) B_{1/2}(t_2) \rangle = \min\{t_1 ; t_2\}$ oBm (Wiener)

4) Correlation of increments

$$\langle [B_H(t) - B_H(t-\tau)][B_H(t+\tau) - B_H(t)] \rangle \xRightarrow{1)-3)}$$

$$\Rightarrow \begin{cases} > 0, & 1/2 < H \leq 1 \quad \text{persistent motion} \\ = 0; & H = 1/2 \\ < 0; & 0 < H < 1/2 \quad \text{antipersistent} \end{cases}$$

5) FBM as moving average (Mandelbrot & van Ness 1968)
Explicit form of FBM

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^{0} \left[(t-t')^{H-1/2} - (-t')^{H-1/2} \right] dB(t') + \int_0^t (t-t')^{H-1/2} dB(t') \right\}, \qquad (*)$$

$0 < H < 1$

Remark 1. $B(t) \equiv B_{1/2}(t) \equiv W(t)$

$dB(t) = \xi(t)\,dt$

$\uparrow$ wGn, $\langle \xi \rangle = 0$, $\langle \xi(t_1)\xi(t_2)\rangle = \delta(t_1 - t_2)$

Remark 2. Prefactor $1/\Gamma(H+1/2)$ is not of importance (added just for historical reasons)

Remark 3. Resembles moving average (math and data analysis) or linear response (phys)

$$f(t) = \int_{-\infty}^{t} K(t-t') g(t')\,dt', \quad f(t) = \int_0^t K(t-t') g(t')\,dt'$$

Remark 4. Motivation of M&vN

- economic time series: strong interdependence between distant samples
- extremely long interdependence encountered in hydrology (Hurst), see Jens Feder, Fractals
- $1/f$ noises (flicker noises) in solids

Exercise. Check Properties 1) - 3) with the use of $(*)$.

8.3 Fractional Gaussian noise (fGn, FGN, ...) as an "exact" derivation of FBM.

Reminder. We had before

$$\frac{dW}{dt} = \frac{dB}{dt} = \xi(t) \text{ white Gaussian noise}$$

$$\frac{dL_\alpha}{dt} = \eta_\alpha(t) \text{ white Lévy noise}$$

Formally $\dfrac{dB_H}{dt} = \xi_H(t)$, $\quad \langle \xi_H(t) \rangle = 0$

Take $\langle B_H(t_1) B_H(t_2) \rangle = \dfrac{\overset{\swarrow \text{const}}{V_H}}{2}\left[t_1^{2H} + t_2^{2H} - |t_2 - t_1|^{2H} \right]$

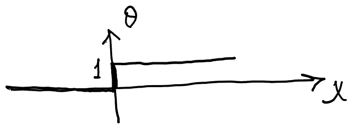
$$\langle \xi_H(t_1)\xi_H(t_2)\rangle = \frac{d^2}{dt_1 dt_2}\langle B_H(t_1) B_H(t_2)\rangle =$$

$$= -\frac{V_H}{2}\frac{d^2}{dt_1 dt_2}|t_1 - t_2|^{2H} \underset{\substack{\uparrow \\ \tau = t_1 - t_2}}{=} \frac{V_H}{2}\frac{d^2}{d\tau^2}|\tau|^{2H} =$$

$$= V_H H \frac{d}{d\tau}\left[|\tau|^{2H-1}\frac{d}{d\tau}|\tau| \right] =$$

$$= V_H H \left[(2H-1)|\tau|^{2H-2}\left(\frac{d}{d\tau}|\tau|\right)^2 + |\tau|^{2H-1}\frac{d^2}{d\tau^2}|\tau| \right] =$$

$$\Rightarrow \quad \frac{d}{dx}|x| = \operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases},$$



$$\operatorname{sgn} x = \theta(x) - \theta(-x)$$

↑ Heaviside function

$$\frac{d}{dx}\operatorname{sgn} x = \frac{d}{dx}\theta(x) - \frac{d}{dx}\theta(-x) = 2\delta(x) \quad \Rightarrow$$

$$\Rightarrow V_H H(2H-1)|\tau|^{2H-2} + 2V_H H|\tau|^{2H-1}\delta(\tau) \qquad f(x)\delta(x) = f(0)$$

$$\langle \xi_H(t_1)\xi_H(t_2)\rangle = \begin{cases} V_H H(2H-1)|\tau|^{2H-2}, & 1/2 < H < 1 \\ V_H\,\delta(\tau) & H = 1/2 \\ ??? & 0 < H < 1/2 \end{cases}$$

↑ the term with $\delta(\tau)$ is not defined

(8.4) fGn via smoothering FBM (M&vN)

$$B_H(t;\delta) = \frac{1}{\delta}\int_t^{t+\delta} B_H(t')\,dt' = \int_{-\infty}^{\infty} B_H(t')\,\varphi(t-t')\,dt'$$

$$\varphi(t) = \begin{cases} \delta^{-1} & \text{if } 0 \le t < \delta \\ 0, & \text{otherwise} \end{cases}$$

δ = infinitesimally small time scale

Introduce $\quad B_H'(t;\delta) = \dfrac{B_H(t+\delta;\delta) - B_H(t;\delta)}{\delta}$

ACF of fGn $\langle \xi_H(t)\,\xi_H(t+\tau)\rangle =$

$$= \frac{1}{\delta^2}\left\langle \left[B_H(t+\delta) - B_H(t)\right]\left[B_H(t+\tau+\delta) - B_H(t+\tau)\right]\right\rangle$$

Exercise (use properties 1) - 3))

1) ACF $\langle \xi_H(t)\,\xi_H(t+\tau)\rangle =$

$$(***) = \frac{V_H}{2\delta^{2-2H}}\left[\left(\frac{|\tau|}{\delta}+1\right)^{2H} + \left|\frac{|\tau|}{\delta}-1\right|^{2H} - 2\left(\frac{|\tau|}{\delta}\right)^{2H}\right]$$

2) Asymptotics at $|\tau|/\delta \gg 1$

$$\underbrace{\langle \xi_H(t)\,\xi_H(t+\tau)\rangle}_{\equiv \langle \xi_H^2\rangle_\tau} \simeq V_H\, H(2H-1)\,|\tau|^{2H-2} = \begin{cases} >0, & \frac{1}{2}<H<1 \\ <0, & 0<H<1/2 \end{cases}$$

3) behavior at short τ

$$\mathrm{Var}\{\xi_H\} = V_H\,\delta^{2H-2}$$

$$\langle \xi_H^2\rangle_\tau \simeq \mathrm{Var}\{\xi_H\} - \frac{V_H}{\delta^2}\tau^{2H}$$

(8.5) Overall behavior of ACF

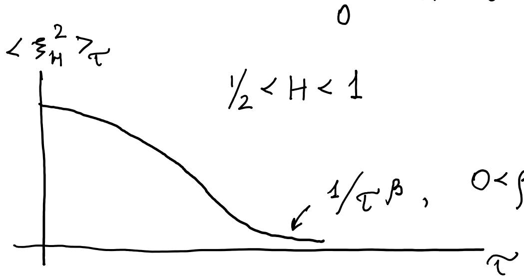
- persistent $\quad 1/2 < H < 1$, $\quad \langle \xi_H^2\rangle_\tau$ positive and finite

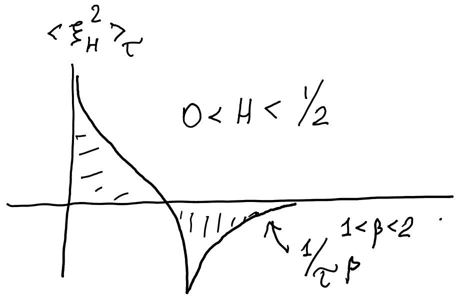
  and $\displaystyle\int_0^{\infty} \langle \xi_H^2\rangle_\tau\, d\tau = \infty \quad \left\{ \int^{\infty} \frac{1}{\tau^{2-2H}}\,d\tau = \infty \right.$

  $0 < 2-2H < 1$

• antipersistent $0 < H < 1/2$. Analysis of (***) says that $\langle \xi_H^2 \rangle_\tau$ changes sign once from positive to negative, and

$$\int_0^\infty \langle \xi_H^2 \rangle_\tau \, d\tau = 0$$





(8.6) Spectral density of fGn

Wiener-Khinchine:

$$\langle \xi_H^2 \rangle_\omega = \int_{-\infty}^{\infty} \langle \xi_H^2 \rangle_\tau e^{i\omega\tau} d\tau$$

$$\Updownarrow$$

$$\langle \xi_H^2 \rangle_\tau = \int_{-\infty}^{\infty} \langle \xi_H^2 \rangle_\omega e^{-i\omega\tau} \frac{d\omega}{2\pi}$$

$\Rightarrow$ Problem $\left\{ \int_0^\infty \langle \xi_H^2 \rangle_\tau d\tau = \infty \right.$ for $1/2 < H < 1$ non-integrable

Formally

$$\langle \xi_H^2 \rangle_\omega = 2 \int_0^\infty d\tau \langle \xi_H^2 \rangle_\tau \cos\omega\tau \sim \int_0^\infty d\tau \frac{\cos\omega\tau}{\tau^{2-2H}} =$$

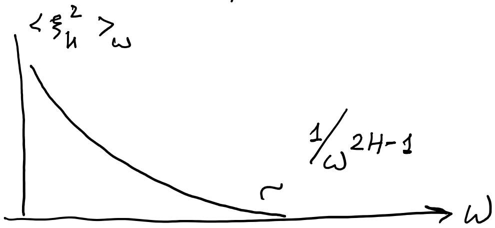
$$= \{ y = \omega\tau \} = \omega^{1-2H} \underbrace{\int_0^\infty dy \frac{\cos y}{y^{2-2H}}}_{} \sim \frac{1}{\omega^\alpha}, \quad \alpha = 2H-1$$

diverges at $y=0$, if $0 < H < 1/2$

Converges at $y = 0$ if $\frac{1}{2} < H < 1$

Converges at $y = \infty$ because of $\cos y$

Persistent case $1/2 < H < 1$



flicker noise ubiquitous in solid state phys

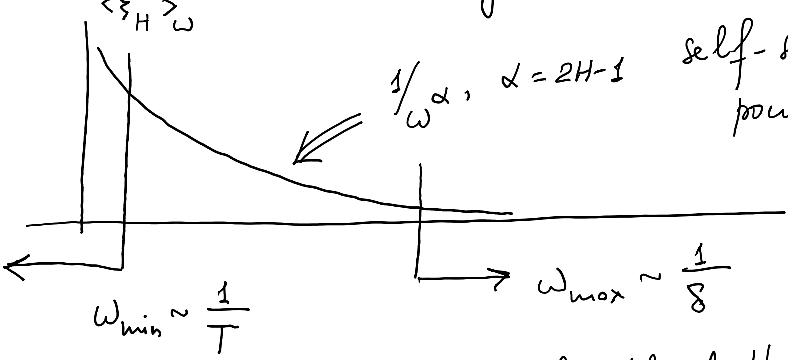
Antipersistent case, $0 < H < 1/2$

How to overcome divergencies?

Mathematically: FT considered in the sense of generalized functions

Physically: Truncation of ACF at short and long times $\Leftrightarrow$ truncation of spectral density at high and low frequencies

Reminder: Uncertainty rule (principle)



T measurement time (length of the trajectory)

δ infinitesimally small time scale (e.g., measuring device resolution, time step in discrete time measurements)

Remark. Can be also physical processes leading to the restrictions on the power-low behavior

e.g., $\text{ACF} \sim \frac{1}{\tau^{2-2H}} \times e^{-\tau/\tau_*} \Rightarrow \begin{cases} \sim 1/\tau^{2-2H} & \tau \ll \tau_* \\ \sim e^{-\tau/\tau_*} & \tau \gg \tau_* \end{cases}$

exponential decay at long times

8.8 Physical picture

$$\frac{dx}{dt} = v(t)$$

random velocity: stationary process

$$\langle v(t) \rangle = 0 \ , \quad \langle v(t_1) v(t_2) \rangle = \langle v^2 \rangle_{\tau = t_2 - t_1}$$

$x(0) = 0$

$\langle x(t) \rangle = 0$

$$\text{MSD} \equiv \langle x^2(t) \rangle = \left\langle \int_0^t v(t_1)\, dt_1 \int_0^t v(t_2)\, dt_2 \right\rangle =$$

$$= \int_0^t dt_1 \int_0^t dt_2 \langle v^2 \rangle_{t_2 - t_1} \underset{\substack{\uparrow\ \tau = t_2 - t_1,\ \langle v^2 \rangle_{|\tau|} \\ \text{lecture (ergodicity)}}}{=} 2 \int_0^t (t - \tau) \langle v^2 \rangle_\tau \, d\tau \quad (*)$$

exact formula

Assumption. $\langle v^2 \rangle_\tau$ decays fast enough at $\tau \to \infty$

$$\langle x^2(t) \rangle \underset{t \to \infty}{\simeq} 2t \underbrace{\int_0^\infty \langle v^2 \rangle_\tau \, d\tau}_{\text{finite}} - 2 \underbrace{\int_0^\infty \tau \langle v^2 \rangle_\tau \, d\tau}_{\text{finite}} \to$$

$$\to 2t \int_0^\infty \langle v^2 \rangle_\tau \, d\tau = 2\underline{\underline{D}}t \ , \quad t \to \infty \ , \quad D = \int_0^\infty \langle v^2 \rangle_\tau \, d\tau$$

(Kubo formula)

Example. $\langle v^2 \rangle_\tau = \langle v^2 \rangle e^{-\tau/\tau_{cor}}$

↳ variance

$$\int_0^\infty \langle v^2 \rangle_\tau \, d\tau = \langle v^2 \rangle \tau_{cor}$$

$$\int_0^\infty \tau \langle v^2 \rangle_\tau \, d\tau = \langle v^2 \rangle \tau_{cor}^2$$

Us. $(*)$ $\Rightarrow$ $\langle x^2(t) \rangle \simeq 2Dt$, $D = \langle v^2 \rangle \tau_{cor}$, $t \gg \tau_{cor}$

normal diffusion

$$\langle x^2(t) \rangle \propto t^1$$

Q. When do we expect anomalous diffusion?

Naively,

$$\int_0^\infty \langle v^2 \rangle_\tau \, d\tau = \begin{cases} \infty, & \text{fast diff (super) } MSD \propto t^\mu, \ \mu > 1 \\ 0, & \text{slow diff (sub) } MSD \propto t^\mu, \ \mu < 1 \end{cases}$$

Compare with persistent ($1/2 < H \le 1$) and antipersistent ($0 < H < 1/2$) cases!

Indeed, $\frac{dB_H}{dt} = \xi_H(t)$, $\langle B_H^2(t) \rangle \propto t^{2H}$, $0 < H \le 1$

$1/2 < H \le 1$ $\quad \int_0^\infty \langle \xi_H^2 \rangle_\tau \, d\tau = \infty$

$0 < H < 1/2$ $\quad \int_0^\infty \langle \xi_H^2 \rangle_\tau \, d\tau = 0$

complete analogy with $\frac{dx}{dt} = v(t)$

Example. Persistent case.

Let $\langle v^2 \rangle_\tau = \frac{D_H}{\Gamma(2H-1)} |\tau|^{2H-2}$, $\frac{1}{2} < H < 1$ (**)

- $[D_H] = \frac{cm^2}{sec^{2H}} \Rightarrow [\langle v^2 \rangle_\tau] = \frac{cm^2}{sec^2}$
- $\int_0^\infty \langle v^2 \rangle_\tau \, d\tau = \infty$ divergence at $\tau = \infty$
- $\Gamma(2H-1)$ captures white noise limit $H = 1/2$

Use $\boxed{\lim_{\alpha \to 0^+} \frac{|\tau|^{\alpha - 1}}{\Gamma(\alpha)} = 2\delta(\tau)}$ (A)

Let $\alpha = 2H - 1$

$$\lim_{H \to 0.5^+} \frac{|\tau|^{2H-2}}{\Gamma(2H-1)} = 2\delta(\tau) \Rightarrow \langle v^2 \rangle_\tau = 2D\delta(\tau), \ D_{1/2} \equiv D$$

- white noise limit: plug (**) into (*)

$$\langle x^2(t) \rangle = 4Dt \underbrace{\int_0^t \delta(\tau) \, d\tau} - 2D \underbrace{\int_0^t \tau \delta(\tau) \, d\tau} = 2Dt$$

$$\langle x^2(t)\rangle = 4Dt \underbrace{\int_0 \delta(\tau)\,d\tau}_{=1/2} - 2D \underbrace{\int_0 \tau\,\delta(\tau)\,d\tau}_{=0}$$

- persistent case: plug (**) into (*)

$$\langle x^2(t)\rangle = \frac{2D_H}{\Gamma(2H-1)} \int_0^t d\tau\,(t-\tau)\,\tau^{2H-2} =$$

$$= \frac{2D_H}{\Gamma(2H-1)} \left\{ t\,\frac{\tau^{2H-1}}{2H-1}\Big|_0^t - \frac{\tau^{2H}}{2H}\Big|_0^t \right\} =$$

$$= \frac{2D_H}{\Gamma(2H-1)} \left\{ \frac{t^{2H}}{2H-1} - \frac{t^{2H}}{2H} \right\} = \frac{2D_H\, t^{2H}}{2H(2H-1)\Gamma(2H-1)} \Rightarrow$$

$z\Gamma(z) = \Gamma(z+1)$

$$\Rightarrow \boxed{\langle x^2(t)\rangle = \frac{2D_H\, t^{2H}}{\Gamma(2H+1)}} \quad \text{superdiff} \quad \tfrac{1}{2} < H \le 1$$

Check. $H = 1/2$ $\langle x^2(t)\rangle = 2Dt$, $D_{1/2} \equiv D$ ok.

Proof of (A). Go to FT

rhs $\displaystyle\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, 2\,\delta(\tau) = \underline{2}$

lhs $\displaystyle\lim_{\alpha\to 0^+} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, |\tau|^{\alpha-1} =$

$$= \lim_{\alpha\to 0^+} \frac{2}{\Gamma(\alpha)} \int_0^\infty d\tau\, \tau^{\alpha-1} \cos(\omega\tau) = \{\eta = |\omega|\tau\} =$$

$$= \lim_{\alpha\to 0^+} \frac{2}{\Gamma(\alpha)|\omega|^\alpha} \underbrace{\int_0^\infty d\eta\, \eta^{\alpha-1}\cos\eta}_{} =$$

$$= \lim_{\alpha\to 0^+} \frac{2}{\cancel{\Gamma(\alpha)}|\omega|^\alpha}\, \cancel{\Gamma(\alpha)} \cos\left(\frac{\pi\alpha}{2}\right) = \underline{2}$$