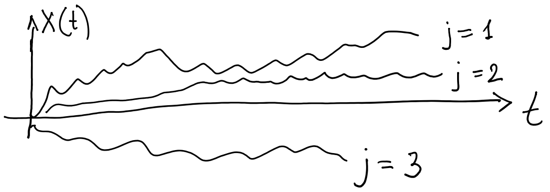
Reminder 1. Stochastic Processes

$X = f(t)$ random function of time



$j = 1, \dots, N,$
$N \gg 1$

$Y = f(\vec{r})$ random function of coordinates
$\shortparallel$
Random field (random surfece $Z = f(x, y)$)

Stoch Proc fully characterized by n-point PDFs

- $p_n(x_1, t_1; x_2, t_2; \dots, x_n, t_n) \quad \forall n$
- n-point moments, e.g.
$$\langle x(t_1) x(t_2) \rangle = \iint_{-\infty}^{\infty} x_1 x_2 \, p_2(x_1, t_1; x_2, t_2) \, dx_1 dx_2$$
- n-point characteristic functions
$$\hat{p}_n(k_1, t_1; k_2, t_2; \dots; k_n, t_n) = \int \dots \int e^{ik_1 x_1 + ik_2 x_2 + \dots + ik_n x_n} \times$$
$$\times \, p_n(x_1, t_1; \dots; x_n, t_n) \, dx_1 \dots dx_n$$

Fortunctely, for most practical purposes enough to know $n = 1, 2$.

Reminder 2 Important classes of stoch Proc.

Stationary processes

strictly stationary
$$p_n(x_1, t+\tau; \dots; x_n, t+\tau) = $$
$$= p_n(x_1, t_1; \dots; x_n, t_n)$$

wide sense stationary
$$\langle x(t) \rangle = const$$
$$\langle x(t_1) x(t_2) \rangle = K(t_1 - t_2)$$

Ergodic Processes

all ensemble averages = corresponding time averages at $T \to \infty$

weakly ergodic (ergodicity in wide sense)
$$\langle x(t) \rangle = \overline{x^{(j)}(t)} \equiv \frac{1}{T} \int_0^T x(t) \, dt$$
$$\langle x(t) x(t+\tau) \rangle = \overline{x^{(j)}(t) \, x^{(j)}(t+\tau)} =$$

$$\langle x(t)x(t+\tau)\rangle = x^{(j)}(t)\,x^{(j)}(t+\tau) \; := $$

$$= \frac{1}{T-\tau}\int_0^{T-\tau} x^{(j)}(t)\,x^{(j)}(t+\tau)\,dt,$$

$$T \to \infty$$

<u>Theorem</u>. Only stationary processes are ergodic

<u>Fact</u>. "Most" of stationary processes are ergodic

Great advantage: enough to have only a single realization (trajectory)

All other processes $\Rightarrow$ Non-stationary processes

<u>Non-stat with independent increments</u>

Sequence $t_1 < t_2 < \ldots < t_n$

$x(t_1),\ x(t_2)-x(t_1),\ x(t_3)-x(t_2),\ \ldots,\ x(t_n)-x(t_{n-1})$

statistically independent

Fully characterized by

$$p(x,t) \doteqdot \hat{p}(k,t) = \langle e^{ikx(t)} \rangle$$

$$p(x_2-x_1;t_1,t_2) \doteqdot \langle e^{ik(x(t_2)-x(t_1))} \rangle$$

Imp. Example. Gaussian process with ind. increm.

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}}\exp\left[-\frac{(x-\mu(t))^2}{2\sigma^2(t)}\right]$$

$$p(x_2-x_1;t_1,t_2) = \frac{1}{\sqrt{2\pi\left(\sigma^2(t_2)-\sigma^2(t_1)\right)}} \times$$

$$\times \exp\left[-\frac{(x_2-x_1-\mu(t_2)+\mu(t_1))^2}{2\left(\sigma^2(t_2)-\sigma^2(t_1)\right)}\right]$$

where $\mu(t) = \langle x(t) \rangle$

$$\sigma^2(t) = \mathrm{Var}[x(t)] \equiv \langle x^2(t)\rangle - \langle x(t)\rangle^2$$

Markov processes (next term)

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Non-stat. with stationary increments

- $x(t+\tau) - x(t)$ stationary in t
  stat. properties of $x(t_2) - x(t_1)$ depends on $t_2 - t_1$, but not on t_2 or t_1 separately
- structure function $S(\tau) = \langle (x(t+\tau) - x(t))^2 \rangle$
  Important example $S(\tau) = \text{const} \cdot \tau^{2H}$, $0 < H \leq 1$
  Fractional Brownian motion

Non-stat. with stationary & independent increments

(Lévy processes)

$$\hat{p}(k,t) = e^{t\Psi(k)}$$

Lévy-Khinchine formula establishes explicit form of $\Psi(k)$.

Example. Wiener process $\Psi(k) = -\frac{1}{2}\sigma^2 k^2$

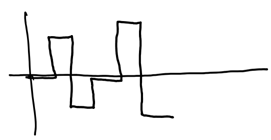
Ordinary BM with linear drift

$$\Psi(k) = ik\mu - \frac{1}{2}\sigma^2 k^2$$

- random process with independent jumps
  (continuous analogue of sums of iid r.v.)

$$x(t_n) = x(t_1) + \sum_{j=2}^{n} \left[x(t_j) - x(t_{j-1}) \right]$$

jumps $\xi_j = \xi(t_j + 0) - \xi(t_j - 0)$



$$x(t) = \sum_{t_j < t} \xi_j \ , \quad \Delta t = t_{j+1} - t_j$$

Look at previous lecture

$$\langle \xi \rangle = 0 \ , \quad \langle \xi^2 \rangle = \sigma^2 < \infty$$

Due to CLT $\Rightarrow$ $x(t)$ is Wiener process

Due to CLT $\Rightarrow$ $x(t)$ is Wiener process

$$p(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

Mean squared displacement $\langle x^2 \rangle = 2Dt$, $D = \frac{\sigma^2}{2\Delta t}$

- White Gaussian noise

formally $\frac{dW}{dt} = \xi(t) = \sqrt{2D}\,\eta(t)$

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t')$$

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \delta(t-t')$$

End of reminder

5.5 α - stable Lévy processes

a) Reminder. Symmetric α-stable laws

- $X, X_1, X_2, \dots, X_n$ iid strictly stable

$$\sum_{j=1}^{n} X_j \overset{d}{=} n^{1/\alpha} X, \quad 0 < \alpha \leq 2$$

- PDF $\hat{p}_X(k) = e^{-\sigma^\alpha |k|^\alpha}$

Particular cases: $\alpha = 2$ $p_X(x) = \frac{e^{-x^2/4\sigma^2}}{\sqrt{4\pi\sigma^2}}$

Cauchy $\alpha = 1$ $\quad p_X(x) = \frac{\sigma}{\pi(x^2+\sigma^2)}$

Notation $p_\alpha(x) \div \hat{p}_\alpha(k)$

- asymptotics $p_\alpha(x) \sim \frac{C(\alpha)\sigma^\alpha}{|x|^{1+\alpha}}$, $C(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p_\alpha(x)\,dx = \infty$$

• GCLT $X_1, \dots, X_n$ iid non-stable, $p(x) \sim \dfrac{A_\alpha}{|x|^{1+\alpha}}$

$0 < \alpha < 2$, A_α const

Properly normalized sum

$$Y_n = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} X_j \Rightarrow p_{Y_n}(y) \xrightarrow[n\to\infty]{} p_\alpha(y) \sim \frac{A_\alpha}{|y|^{1+\alpha}}$$

$$p_\alpha(y) \div \hat{p}_\alpha(k) = e^{-\sigma_Y^\alpha |k|^\alpha} \Rightarrow$$

$$\Rightarrow \sigma_Y = \left[\frac{\pi A_\alpha}{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)} \right]^{1/\alpha}$$

End of reminder

b) random walk approximation to α-stable processes

Random walk: succession of random steps (jumps)

$\to$ setup $t_1 = 0$, $t_n = t$, $\Delta t = \dfrac{t}{n}$, $x(t) = \sum_{j=1}^{n} \xi_j$

ξ_j α-stable iid $\hat{p}_\alpha(k) = \langle e^{ik\xi_j} \rangle = e^{-\sigma^\alpha |k|^\alpha}$.

$$p(x,t) \div \hat{p}(k,t) = \langle e^{ikx(t)} \rangle = \langle e^{ik\sum_{j=1}^{n}\xi_j} \rangle =$$

$$= \prod_{j=1}^{n} \langle e^{ik\xi_j} \rangle = e^{-\sigma^\alpha |k|^\alpha n} = e^{-\sigma^\alpha |k|^\alpha \frac{t}{\Delta t}} =$$

$$= e^{-D_\alpha |k|^\alpha t}, \quad D_\alpha = \frac{\sigma^\alpha}{\Delta t}$$

PDF of α-stable process (symmetric)

$$\boxed{p_\alpha(x,t) \div \hat{p}_\alpha(k,t) = e^{-D_\alpha |k|^\alpha t}}, \quad 0 < \alpha \leq 2$$

Particular examples.

1) Wiener process $\alpha = 2$ $\quad p_{\alpha=2}(x,t) = \dfrac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi D t}}$, $D \equiv D_2$

2) Cauchy process $\quad p_{\alpha=1}(x,t) = \dots$

c) α-stable process as a random walk via GCLT

$\rightarrow t_1 = 0,\ t_n = t,\ \Delta t = \frac{t}{n},\ x(t) = \sum_{j=1}^{n} \xi_j$

ξ_j not α-stable iid, but $p_\xi(x)$ belongs to the domain of attraction of (symmetric) α-stable law, $p_\xi(x) \sim \frac{A_\alpha}{|x|^{1+\alpha}}$

$\Rightarrow$ GCLT $\quad Z = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} \xi_j$

$$p_Z(z) \xrightarrow[n\to\infty]{} p_\alpha(x) \sim \frac{A_\alpha}{|x|^{1+\alpha}},\quad \hat{p}_\alpha(k) = e^{-\sigma_Z^\alpha |k|^\alpha}$$

$\Rightarrow x(t):\ p(x,t) \div \hat{p}(k,t) = \langle e^{ikx(t)} \rangle =$

$$= \langle e^{ik\sum_{j=1}^{n}\xi_j} \rangle = \langle e^{ikn^{1/\alpha} Z} \rangle \xrightarrow[n\to\infty]{} e^{-\sigma_Z^\alpha |k|^\alpha n} =$$

$$= e^{-D_\alpha |k|^\alpha t}$$

$$D_\alpha = \frac{\sigma_Z^\alpha}{\Delta t} = \frac{1}{\Delta t}\left[\frac{\pi A_\alpha}{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)}\right]^{1/\alpha}$$

Conclusion. Iid random jumps (steps) belong to domain of attraction of α-stable law $\Rightarrow$

$\Rightarrow$ after many steps the resulting random walk can be viewed as α-stable Lévy process

d) Important summary

Random walks with "short" and "long" jumps

- short jumps: $\langle \xi^2 \rangle$ finite $\Rightarrow$ random walk $\rightarrow$

$\xrightarrow[n\to\infty]{}$ Wiener process (CLT)

$\xrightarrow[n\to\infty]{}$ Wiener process (CLT)

- long jumps : $\langle \xi^2 \rangle$ infinite $\Rightarrow$
  random walk $\xrightarrow[n\to\infty]{}$ α-stable process (GCLT)

$$\boxed{\text{Lévy flights} \equiv \text{random walk with long jumps}}$$

e) self-similarity of LFs (see self-sim. Wiener)

$$\mathcal{L}_\alpha(\lambda t) \overset{d}{=} \lambda^H \mathcal{L}_\alpha(t) \qquad H \text{ Hurst index } (*)$$

Use $\langle e^{ik\mathcal{L}_\alpha(t)} \rangle = e^{-D_\alpha |k|^\alpha t}$

$$\left.\begin{array}{l} \text{lhs}(*) \quad \langle e^{ik\mathcal{L}_\alpha(\lambda t)} \rangle = e^{-D_\alpha |k|^\alpha \lambda t} \\ \text{rhs}(*) \quad \langle e^{i\underbrace{k\lambda^H}\mathcal{L}_\alpha(t)} \rangle = e^{-D_\alpha |k|^\alpha \lambda^{H\alpha} t} \end{array}\right\} \boxed{H = \frac{1}{\alpha}}$$

$\alpha = 2$ Wiener $H = 1/2$

f) Asymptotics, scaling variable, moments of LFs

- Asymptotics
  - $\rightarrow$ α-stable rv X $\quad p_X(x) \equiv p_\alpha(x) \div \hat{p}_\alpha(k) = e^{-\sigma^\alpha |k|^\alpha}$

$$p_\alpha(x) \underset{x\to\infty}{\sim} \frac{C(\alpha)\sigma^\alpha}{|x|^{1+\alpha}}, \quad C(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)$$

$\rightarrow$ α-stable symmetric random process $\mathcal{L}_\alpha(t)$

$$p_\alpha(x,t) \div \hat{p}_\alpha(k,t) = e^{-D_\alpha |k|^\alpha t}$$

$$p_\alpha(x,t) \sim C(\alpha)\frac{D_\alpha t}{|x|^{1+\alpha}}, \quad 0 < \alpha < 2$$

• Scaling variable

$$p_\alpha(x,t) = \int_{-\infty}^{\infty} e^{-ikx}\,\hat{p}_\alpha(k,t)\frac{dk}{2\pi} = \int_{-\infty}^{\infty} e^{-ikx - D_\alpha|k|^\alpha t}\frac{dk}{2\pi} =$$

$$= \left\{ k' = (Dt)^{1/\alpha} k \right\} = \frac{1}{(Dt)^{1/\alpha}} \int_{-\infty}^{\infty} e^{-ik'x' - |k'|^\alpha} \frac{dk'}{2\pi} =$$

$$= \left\{ x' = \frac{x}{(D_\alpha t)^{1/\alpha}} \right\} = \frac{1}{(Dt)^{1/\alpha}}\, p_\alpha(x') \Big|_{\sigma=1}$$

$$\boxed{p_\alpha(x,t) = \frac{1}{(D_\alpha t)^{1/\alpha}}\, p_\alpha\left(\frac{x}{(D_\alpha t)^{1/\alpha}}\right), \quad \hat{p}_\alpha(k) = e^{-|k|^\alpha}, \quad 0<\alpha\leq 2}$$

Example. Wiener process $p_\alpha(x,t) \to p_{\alpha=2}(x,t) =$

$$= \frac{1}{(Dt)^{1/\alpha}}\, g\left(\frac{x}{(Dt)^{1/\alpha}}\right), \quad \hat{g}(k) = e^{-k^2}$$

$$g(x) = \int_{-\infty}^{\infty} e^{-ikx-k^2}\frac{dk}{2\pi} = \frac{e^{-x^2/4}}{\sqrt{4\pi}} \Rightarrow$$

$$\Rightarrow p_{\alpha=2}(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

• q-th moments

$$\langle x^2(t)\rangle = \int_{-\infty}^{\infty} x^2 p_\alpha(x,t)\,dx = \infty$$

$$\langle |x(t)|^q\rangle = 2\int_0^\infty x^q p_\alpha(x,t)\,dx = 2\int_0^\infty x^q \frac{1}{(Dt)^{1/\alpha}}\, p_\alpha\left(\frac{x}{(D_\alpha t)^{1/\alpha}}\right)dx =$$

$$= \left\{ y = \frac{x}{(Dt)^{1/\alpha}} \right\} = 2\underline{\underline{(Dt)^{q/\alpha}}} \int_0^\infty y^q p_\alpha(y)\,dy, \quad \hat{p}_\alpha(k) = e^{-|k|^\alpha}$$

$$p_\alpha(y) \sim \frac{1}{y^{1+\alpha}} \Rightarrow \int_0^\infty y^q p_\alpha(y)\,dy = C(q,\alpha) < \infty$$

if $q<\alpha$

$$p_\alpha(y) \sim \frac{1}{y^{1+\alpha}} \Rightarrow \int_0 y^q\, p_\alpha(y)\, dy = C(q,\alpha) < \infty$$
if $q<\alpha$

$$\langle |x(t)|^q \rangle \sim \begin{cases} t^{q/\alpha}, & q<\alpha \\ \infty, & q \geq \alpha \end{cases}, \quad 0<\alpha<2$$

Important summary

$\alpha = 2$ Gauss $\langle |x|^q \rangle \sim t^{qH}$, $H = 1/2$ $\forall q$

$0<\alpha<2$ $\langle |x|^q \rangle \sim t^{qH}$, $H = 1/\alpha$, $0<q<\alpha$

Remark. This is direct consequence of self-similarity

$$x(\lambda t) \stackrel{d}{=} \lambda^H x(t), \quad \lambda \to t, \quad t=1$$

$$\langle |x(t)|^q \rangle \sim \langle |t^H x(1)|^q \rangle \sim t^{qH} \langle |x(1)|^q \rangle$$

Terminology. $\langle |x(t)|^q \rangle \sim t^{f(q)}$

$f(q) = \text{const} \cdot q$ self-similar or monofractal random process

$f(q)$ nonlinear random multifractal but $f(q)$ not arbitrary!

g) white Lévy noise (α-stable)

$$\frac{d\mathcal{L}_\alpha(t)}{dt} = \xi_\alpha(t), \quad \mathcal{L}_\alpha(t) = \int_0^t \xi_\alpha(t')\,dt'$$

$$\mathcal{L}_\alpha(t+\Delta t) - \mathcal{L}_\alpha(t) = \mathcal{L}(\Delta t) - \mathcal{L}_\alpha(0) = (\Delta t)^{1/\alpha} \mathcal{L}_\alpha(1) \quad (**)$$

$$\mathcal{L}_\alpha(t) : \hat{p}_\alpha(k,t) = e^{-D_\alpha |k|^\alpha t}$$

Denote $\mathcal{L}_\alpha(1) \equiv \xi_\alpha$: $\hat{p}_{\xi_\alpha}(k) = e^{-D_\alpha |k|^\alpha}$

Let $\xi_\alpha = D_\alpha^{1/\alpha} \eta_\alpha \Rightarrow \hat{p}_{\eta_\alpha}(k) = e^{-|k|^\alpha}$

standard symmetric α-stable distribution

Back to $(**)$

$$\mathcal{L}_\alpha(t+\Delta t) - \mathcal{L}_\alpha(t) = (D\Delta t)^{1/\alpha}\eta_\alpha \quad , \quad 0<\alpha\leq 2$$

η_α standard SαS variable $\hat{p}_\alpha(k) = e^{-|k|^\alpha}$

<u>Remark</u>. Standard SαS with $\alpha=2$ is $\mathcal{N}(0,2)$

Remember before $\eta \sim \mathcal{N}(0,1)$, $\langle\eta\rangle = 0$, $\langle\eta^2\rangle = 1$

but $\langle\eta_\alpha^2\rangle\big|_{\alpha=2} = 2$!

h) fractional diffusion equation

- Wiener process $\alpha=2$ $\hat{p}_2(k,t) = e^{-Dk^2t}$

$$\left.\begin{aligned} \frac{\partial}{\partial t}\hat{p}_2(k,t) &= -Dk^2\hat{p}_2(k,t) \\ \hat{p}_2(k,t=0) &= 1 \end{aligned}\right\} \Rightarrow \text{eq. for } p_2(x,t)$$

$$\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial\hat{p}_2(k,t)}{\partial t}\frac{dk}{2\pi} = -D\int_{-\infty}^{\infty} e^{-ikx}k^2\hat{p}_2(k,t)\frac{dk}{2\pi}$$

$$\left.\begin{aligned} \frac{\partial p_2(x,t)}{\partial t} &= D\frac{\partial^2}{\partial x^2}p_2(x,t) \\ p_2(x,t=0) &= \delta(x) \end{aligned}\right\}, \quad -\infty<x<\infty$$

Solution $p_2(x,t) = \dfrac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}$

- α-stable , $0<\alpha<2$. $\hat{p}_\alpha(k,t) = e^{-D_\alpha|k|^\alpha t}$

$$\left.\begin{aligned} \frac{\partial}{\partial t}\hat{p}_\alpha(k,t) &= -D_\alpha|k|^\alpha\hat{p}_\alpha(k,t) \\ \hat{p}_\alpha(k,t=0) &= 1 \end{aligned}\right\} \Rightarrow \text{Eq for } p_\alpha(x,t)\ ?$$

<u>Digression</u> Space fractional derivative via its Fourier transform

FT pair $\phi(x) \doteq \hat\phi(k) = \int\limits_{-\infty}^{\infty} e^{ikx}\,\phi(x)\,dx$

$$\phi(x) = \int\limits_{-\infty}^{\infty} e^{-ikx}\,\hat\phi(k)\,\frac{dk}{2\pi}$$

1st derivative $\dfrac{d\phi}{dx} \doteq -ik\hat\phi(k)$

2nd derivative $\dfrac{d^2\phi}{dx^2} \doteq -k^2\hat\phi(k)$

Symmetric Riesz derivative $\dfrac{d^\alpha\phi}{d|x|^\alpha} \doteq -|k|^\alpha\hat\phi(k)$

this is just the notation!

$$\alpha = 2 \qquad \frac{d^2\phi}{d|x|^2} = \int\limits_{-\infty}^{\infty}(-k^2)\hat\phi(k)\,e^{-ikx}\,\frac{dk}{2\pi} = \frac{d^2\phi}{dx^2}$$

But $\dfrac{d\phi}{d|x|} \doteq -|k|\hat\phi(k) \Rightarrow \dfrac{d\phi}{d|x|} \neq \dfrac{d\phi}{dx}$

Example. $\phi(x) = \dfrac{1}{1+x^2} \Rightarrow \dfrac{d^\alpha\phi}{d|x|^\alpha} = -\int\limits_{-\infty}^{\infty}\dfrac{dk}{2\pi}\,e^{-ikx}|k|^\alpha\hat\phi(k) =$

$$= \frac{\Gamma(\alpha+1)}{\pi\,(1+x^2)^{(\alpha+1)/2}}\cos\left[(\alpha+1)\arctan x\right]$$

End of digression (more details next term)

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}p_\alpha(x,t) &= D_\alpha\frac{\partial^\alpha}{\partial|x|^\alpha}p_\alpha(x,t) \\ p_\alpha(x,t=0) &= \delta(x),\ -\infty < x < \infty \end{aligned}}$$

Generalization to $d \geq 1$ via <u>fractional Laplacian</u>

Generalisation to $d \geq 1$ via fractional Laplacian

$$\boxed{\begin{array}{ll} \text{"Normal"} & \dfrac{\partial p}{\partial t} = D \Delta p(\vec{r}, t) \\ \text{Space-Fractional} & \dfrac{\partial}{\partial t} p(\vec{r}, t) = -D(-\Delta)^{\alpha/2} p(\vec{r}, t) \end{array}}$$

Assignment 3

#2 ACF $B(\tau)$

$$\left.\begin{aligned} \tau_{cor} &= \frac{1}{B(0)} \int_0^\infty B(\tau)\, d\tau \\ \Delta\omega &= \frac{1}{G_{max}} \int_0^\infty G(\omega)\, d\omega \end{aligned}\right\} \Delta\omega \cdot \tau_{cor} =$$

Alternatively,

$$\tau_{cor} = \frac{2}{B(0)} \int_0^\infty B(\tau)\, d\tau$$

$$\Delta\omega = \frac{2}{G_{max}} \int_0^\infty G(\omega)\, d\omega$$

In the lecture

$$\left.\begin{aligned} \tau_{cor} &= \frac{2}{B(0)} \int_0^\infty B(\tau)\, d\tau \\ \Delta\omega &= \frac{1}{G_{max}} \int_0^\infty G(\omega)\, d\omega \end{aligned}\right\} \Delta\omega \cdot \tau_{cor} = \pi$$

#3 OK

#4 OK $B_\xi(\tau) = \langle \xi(t)\xi(t+\tau) \rangle =$

$$\equiv \iiint a\cos(\eta t + \varphi)\, a\cos[\eta(t+\tau) + \varphi]\, p_\varphi\, p_\eta\, p_a\, d\varphi\, d\eta\, da =$$

$$= \frac{\langle a^2 \rangle}{2\pi} \iint \cos(\eta t + \varphi)\cos(\eta t + \eta\tau + \varphi)\, p_\eta\, d\varphi\, d\eta =$$

$$= \frac{\langle a^2 \rangle}{2} \int \cos(\eta\tau)\, p_\eta\, d\eta = \frac{\langle a^2 \rangle}{2} \int \frac{e^{i\eta\tau} + e^{-i\eta\tau}}{2} p_\eta\, d\eta =$$

$$= \underbrace{\frac{\langle a^2 \rangle}{4}\left(\hat{p}_\eta(\tau) + \hat{p}_\eta(-\tau)\right)} \neq \frac{\langle a^2 \rangle}{2} p_\eta(\tau) \; !$$

#5. OK. $\xi(t) = a\cos(\omega t + \phi)$, $p(\phi) = \frac{1}{2\pi}$, $0 \le \phi < 2\pi$

$$\hat{p}_\xi(k,t) = \langle e^{ika\cos(\omega t+\phi)} \rangle =$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{ika\cos\psi}, \quad \psi(t) = \omega t + \phi \qquad (1)$$

$$f(\psi) \equiv e^{iz\cos\psi} = \sum_{n=-\infty}^{\infty} A_n(z) e^{in\psi} \qquad (2)$$

$$A_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi) e^{-in\psi}\, d\psi$$

Plug (2) into (1) $\Rightarrow$ $\hat{p}_\xi(k,t) = J_0(ka)$

$$p_\xi(x) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2 - x^2}}, & 0 \le x < a \\ 0, & x \geq a \end{cases}$$

#6. $\xi(t) = a\cos(\omega t + \phi)$, $p_a(x)$, $p(\phi) = \frac{1}{2\pi}$, $0 \le \phi < 2\pi$

$$\hat{p}_\xi(k,t) = \langle e^{ika\cos(\omega t+\phi)} \rangle =$$

$$= \int_{-\infty}^{\infty} dx\, p_a(x) \underbrace{\int_0^{2\pi} \frac{d\phi}{2\pi} e^{ikx\cos(\omega t+\phi)}}_{J_0(kx)} \Rightarrow$$

$$\Rightarrow \hat{p}_\xi(k) = \int_{-\infty}^{\infty} J_0(kx) p_a(x)\, dx$$

Hankel transformation

$$F_\nu(k) = \int_0^\infty J_\nu(kx) f(x)\, x\, dx$$

$$f(x) = \int_0^\infty F_\nu(k) J_\nu(kx)\, k\, dk$$

Used in 2D BM

Go back to # 1. Cauchy is OK.

<u>Hint</u>. Consider Laplace. $p(x) = \frac{1}{2b} e^{-|x|/b} \doteqdot$

$$\hat{p}(k) = \frac{1}{1+b^2k^2}$$

Inf div. $\hat{p}(k) = [\hat{\phi}_n(k)]^n$ for any n

$$\frac{1}{1+b^2k^2} = \left[\left(\frac{1}{1-ibk}\right)^{1/n} \left(\frac{1}{1+ibk}\right)^{1/n}\right]^n =$$

$$= \left[\hat{\phi}_n(k)\right]^n$$

$\hat{\phi}_n(k)$ is the CF of $Y_{1n} - Y_{2n}$,

$\underbrace{\text{two}}_{\text{ind}}$ identically $\underbrace{\text{distributed}}_{\text{(iid)}}$ r.v. with $CF = \frac{1}{(1-ibk)^{1/n}}$

PDF ?

Note $p_X(x) \doteqdot \hat{p}_X(k)$ $\qquad Y = -X \Rightarrow \hat{p}_Y(k) = \hat{p}_X^*(k)$

$p_Y(x) \doteqdot \hat{p}_Y(k)$

End of the hint.