Lecture 16

16 December 2024 10:11

- 6 th January : Bayesian inference (Dr. Saumudrajit Thapa)
- 13 th January (?)
- On the NY eve: Assignment No. 3
- 20th January: lecture
- Assignment No. 4

Written exam ~ 10 th February

II.4. Non-stationary processes with stationary increments

Reminder: Nonstationary processes with independent increments

Two charact. functions fully determine process

$$\hat{p}(k,t) = \langle e^{ikx(t)} \rangle$$

$$\hat{p}(k,t_1,t_2) = \langle e^{ik(x(t_2)-x(t_1))} \rangle$$

$$\Downarrow$$

1-point PDF and 2-point PDF

$$\hat{p}_2(k_1,t_1;k_2,t_2) = \langle e^{ik_1x(t_1)+ik_2x(t_2)} \rangle =$$

$$= \iint_{-\infty}^{\infty} p_2(x_1,t_1;x_2,t_2)\, e^{ik_1x_1+ik_2x_2}\, dx_1 dx_2$$

$$\hat{p}(k,t_1,t_2) = \hat{p}_2(k,t_1;-k,t_2)$$

Remark. Non-stat. processes with ind. incr. $\in$ class of Markov processes (later)

Gaussian processes with independent increments

$$\hat{p}(k,t) = \exp\left[ik\mu(t) - \frac{\sigma^2(t)k^2}{2}\right] \div p(x,t) = \frac{e^{-(x-\mu(t))^2/2\sigma^2(t)}}{\sqrt{2\pi\sigma^2(t)}}$$

$$\langle x(t)\rangle = \mu(t),\quad Var[x(t)] = \langle (x(t)-\langle x(t)\rangle)^2\rangle = \sigma^2(t)$$

$$\hat{p}(k,t_1,t_2) = \exp\left[ik\mu_\Delta(t_1,t_2) - \frac{\sigma_\Delta^2(t_1,t_2)k^2}{2}\right] \leftrightarrow$$

$$\leftrightarrow p(x_1,t_1;x_2,t_2) = p(x_2-x_1,t_1,t_2) = \{\text{see previous lect}$$

$$= \frac{1}{\sqrt{2\pi(\sigma^2(t_2)-\sigma^2(t_1))}} \exp\left[-\frac{(x_2-x_1-(\mu_2(t)-\mu_1(t)))^2}{2(\sigma^2(t_2)-\sigma^2(t_1))}\right]$$

$$t_2 > t_1$$

Particular cases

1. Wiener process (ordinary Brownian motion) (oBm)
$\mu(t) = 0$, $\sigma^2(t) = ct$, $c > 0$
2. Scaled Brownian motion (sBm)
$\mu(t) = 0$, $\sigma^2(t) = ct^\alpha$, $c > 0$, $\alpha > 0$, $\alpha \neq 1$

(4.1) Definition nonstat. process with stat. incr.

- $x(t)$ nonstationary, but
- $x(t+\tau) - x(t)$ stationary in t
(statistical properties of $x(t_2) - x(t_1)$ depend on $t_2 - t_1$), but:

can be non-independent $\Rightarrow$ $\hat{p}(k, t)$ and $\hat{p}(k, t_1, t_2)$ are not enough (in general) to define $x(t)$.

Remark. If $x(t)$ stationary $\Rightarrow$ increments of $x(t)$ are stationary as well

(4.2) Mean value

Remind: for stationary process $\langle x(t) \rangle = \text{const}$

$\langle x(t+\tau) - x(t) \rangle \neq f(t) \Rightarrow$ e.g., $\langle x(t) \rangle = at + b$, $a, b = \text{const}$

Example 1. Wiener process (oBm):
increments independent and stationary
$\sigma^2(t+\tau) - \sigma^2(t) = c\tau$

Example 2. Gaussian with $\sigma^2(t) = ct$,
linear drift $\mu(t) = vt$, $v = \text{const}$ $\Rightarrow$
$\Rightarrow$ increments independent and stationary

Example 3. Scaled Brownian motion (sBm) $\Rightarrow$
$\Rightarrow$ increments independent but nonstationary

Example 4. $x(t) = \alpha t + \xi(t)$
α r.v. $\langle \alpha \rangle = a$, $\xi(t)$ stationary, $\langle \xi(t) \rangle = b$
$\Rightarrow$ increments are stationary

(4.3) Structure function $D(\tau)$

$$x(t) = \langle x(t) \rangle + \underbrace{\xi(t)}_{\text{fluctuations of } x(t)}, \quad \langle \xi(t) \rangle = 0$$

$$D(\tau) = \langle (\xi(t+\tau) - \xi(t))^2 \rangle$$

- Structure function ≡ mean squared increment, MSI
- Analogue of the corr function
- Can be also constructed for stationary process

$$D(\tau) = \langle \xi^2(t+\tau) \rangle - 2\langle \xi(t)\xi(t+\tau) \rangle + \langle \xi^2(t) \rangle =$$

$$= \{ B_\xi(\tau) = \langle \xi(t)\xi(t+\tau) \rangle \} =$$

$$= 2\left[B_\xi(0) - B_\xi(\tau) \right]$$

Remark. $B(\tau) \le B(0)$ because $D(\tau) \geq 0$

- ! Can be evaluated as a time average along a single trajectory

$$\langle (\xi(t+\tau) - \xi(t))^2 \rangle \underset{T\to\infty}{=} \frac{1}{T-\tau} \int_0^{T-\tau} (x(t+\tau) - x(t))^2 dt$$

Example. Random line $x(t) = \alpha t + c$

α r.v., $c = \text{const}$

$$D(\tau) = A\tau^2, \quad A = \langle \alpha^2 \rangle$$

- upper bound for structure function of the process with stationary increments

$$D(\tau) \leq A\tau^2, \quad \tau \uparrow$$

(proved with Minkowski's inequality)

- class of scale-free structure functions (automodality) (Kolmogorov in 1941)

$$D(\tau) = A\tau^{2H}, \quad 0 < H < 1 \quad \text{Hurst exponent (physics lit.)}$$

~~Special symmetry in physical system:~~

scale change of independent variable (time)

Special symmetry in physical system:
scale changes of independent variable (time)
are compensated with similar transformations
of another dynamical variable

$$t \to \lambda t\,, \quad x \to \lambda^{-H} x \implies D(\lambda\tau) = \lambda^{2H} D(\tau)$$

- Definition: self-similar random process or
  random fractal

$$x(\lambda t) \overset{d}{=} \lambda^{H} x(t) \qquad \overset{d}{=} : \text{equality in probability}$$

Check. lhs and rhs have the same structure function.

II.5. Lévy processes: independent and stationary increments
(homogeneous processes with independent increments)

5.1.
$$x(t+\tau) - x(t) \overset{d}{=} x(\tau)$$
$$x(0) = 0$$

Important: PDF charecterized by its CF
(Lévy-Khinchine formula)

$$\hat{p}(k,t) = e^{t\underbrace{\Psi(k)}_{}}$$

L-Kh establish general form of $\Psi(k)$

<u>Example</u>. oBm with linear drift

$$\Psi(k) = ik\mu - \frac{1}{2}\sigma^2 k^2$$

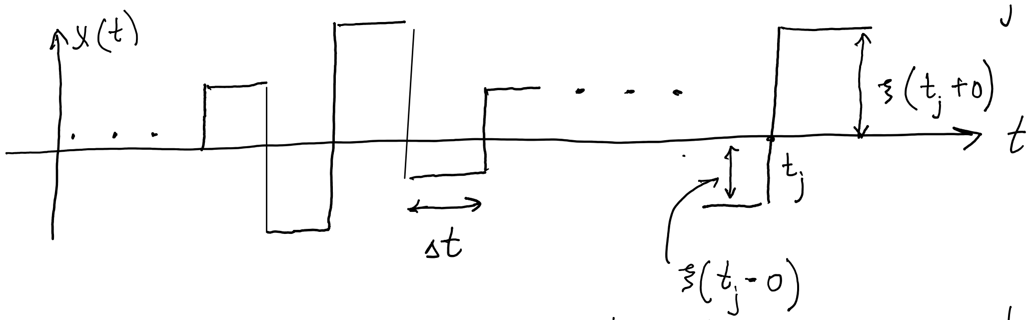
5.2. continuous-time analogue of sums of i.i.d. r.v

$$x(t_n) = x(t_1) + \sum_{j=2}^{n} \underbrace{\left[x(t_j) - x(t_{j-1}) \right]}_{}$$

random, independent,
statistically identical over
time intervals of the same length
(because of their stationarity)

Example. Random process with iid random jumps in fixed time intervals

jumps $\xi_j = \xi(t_j+0) - \xi(t_j-0)$, $x(t) = \sum_{t_j<t} \xi_j$



With this toy example we built two important continuous-time processes

(5.3) Wiener process

a) discrete time representation

setup $t_1 = 0$, $t_n = t$, $\Delta t = \frac{t}{n}$, $x(t) = \sum_{j=1}^{n} \xi_j$

jumps ξ_j Gaussian $\hat{p}_\xi(k) = e^{-\frac{1}{2}\sigma^2 k^2}$

$$\Updownarrow$$

$$p_\xi(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \quad \langle \xi^2 \rangle = \sigma^2, \quad \langle \xi \rangle = 0$$

• PDF $x(t)$: $p(x,t) \doteq \hat{p}(k,t) = [\hat{p}_\xi(k)]^n = e^{-\frac{\sigma^2}{2}k^2 n} =$

$$= \left\{ n = \frac{t}{\Delta t} \right\} = \exp\left[-\tfrac{1}{2}Dk^2 t\right], \quad D = \frac{\sigma^2}{\Delta t}, \quad \Psi(k) = \frac{Dk^2}{2}$$

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{p}(k,t)\,dk = \frac{\exp\left(-\frac{x^2}{2Dt}\right)}{\sqrt{2\pi D t}}$$

Check. $\int_{-\infty}^{\infty} p(x,t)\,dx = 1$, $\langle x(t) \rangle = 0$, $\langle x^2(t) \rangle = Dt$

Conclusion 1. Jumps are Gaussian iid $\Rightarrow$ $\Rightarrow p(x,t)$ Gaussian with $\mathrm{Var}[x(t)] = Dt$

Q. how CLT is applied if $\{\xi_j\}$ are not Gaussian, BUT $\langle \xi \rangle = 0$, $\langle \xi^2 \rangle = \sigma^2 < \infty$ $\Rightarrow$

$\Rightarrow$ $p_\xi(x)$ belongs to the domain of attraction of the Gaussian law

Reminder. CLT Properly normalized sum

$$Z = \frac{1}{\sqrt{n}}\left(\xi_1 + \ldots + \xi_n\right)$$

$$p_Z(z) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right) \doteqdot e^{-\frac{1}{2}\sigma^2 k^2}$$

$$\langle Z^2 \rangle = \sigma^2$$

PDF of $X = \sqrt{n}\, Z$

Reminder. $X = aZ \Rightarrow \hat{p}_X(k) = \hat{p}_Z(ak)$

$$p(x,t) \doteqdot \hat{p}(k,t) \underset{n\to\infty}{=} \hat{p}_Z(\sqrt{n}\,k) = \exp\left(-\frac{\sigma^2}{2}k^2 n\right) = \left\{n = \frac{t}{\Delta t}\right\}$$

$$= \exp\left(-D k^2 t\right), \quad D = \frac{\sigma^2}{2\Delta t}, \quad t \to \infty$$

$$p(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

norm, $\langle x(t) \rangle = 0$, $\langle x^2(t) \rangle = 2Dt$

Conclusion 2. Jumps are non-Gaussian iid with finite variance $\Rightarrow$ after many steps $p(x,t) =$ Gauss with $Var[x(t)] = 2Dt$

Terminology. D diffusion coeff.

$\langle (x(t) - x(0))^2 \rangle$ mean squared displacement, with $x(0) = 0$

$$\overset{\shortparallel}{0}$$

$\Rightarrow$ Wiener process (oBm) appears as the limit one for sum of iid random jumps with finite variance due to CLT $\Rightarrow$ important and ubiquitous (math rigorous formulation : Donsker Theorem) due to CLT $\Rightarrow$ important and ubiquitous

(math rigorous formulation: Donsker Theorem)

$\Rightarrow$ Generic example of <u>random walk</u>: the process can be viewed as a succession of random jumps

<u>Remark</u>. Trajectories of Wiener process are continuous, but nondifferentiable

b) self-similarity of Wiener process $W(t)$

$$W(\lambda t) \overset{d}{=} \lambda^H W(t) \qquad (*) \quad \text{Hurst}$$

Use $\langle e^{ikW(t)} \rangle = e^{-Dk^2 t}$

$$\left.\begin{aligned} &\text{lhs of } (*) \quad \langle e^{ikW(\lambda t)} \rangle = e^{-Dk^2 \lambda t} \\ &\text{rhs of } (*) \quad \langle e^{ik\lambda^H W(t)} \rangle = e^{-D\lambda^{2H} k^2 t} \end{aligned}\right\} \Rightarrow \boxed{H = \frac{1}{2}}$$

c) Mean squared displacement (MSD) and autocorrelation function (ACF)

- $$\langle W^2(t) \rangle = -\left.\frac{\partial^2 \hat{p}(k,t)}{\partial k^2}\right|_{k=0} = 2Dt$$

- $$\langle W(t_1) W(t_2) \rangle \underset{t_1 < t_2}{=} \langle W(t_1) [W(t_1) + W(t_2) - W(t_1)] \rangle =$$

$$= \underbrace{\langle W^2(t_1) \rangle}_{= 2Dt_1} + \underbrace{\langle W(t_1) [W(t_2) - W(t_1)] \rangle}_{= 0 \text{ independence of increments}} = 2Dt_1$$

By analogy $\langle W(t_1) W(t_2) \rangle \underset{t_1 > t_2}{=} \langle W^2(t_2) \rangle = 2Dt_2$

$$\boxed{\langle W(t_1) W(t_2) \rangle = 2D \min\{t_1, t_2\}}$$

- structure function $S(\tau) = \langle (W(t+\tau) - W(t))^2 \rangle = ?$

d) oBm as integral of white noise

<u>Digression</u>. More general case: integral of a stationary process

$$\frac{dx}{dt} = \xi(t), \quad x(t) = \int_0^t \xi(t')dt'$$ stochastic integral

$\xi(t)$ ← stationary process

$$\langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t+\tau) \rangle = B(\tau)$$

<u>Property 1</u>. $x(t)$ has stationary increments.

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

<u>Property 2</u> MSD for integral of stationary process

$$\langle x^2(t) \rangle = \langle \int_0^t \xi(t_1)dt_1 \int_0^t \xi(t_2)dt_2 \rangle =$$

$$= \int_0^t dt_1 \int_0^t dt_2 \langle \xi(t_1)\xi(t_2) \rangle = \int_0^t dt_1 \int_0^t dt_2 \, B(t_2 - t_1) =$$

$= \{$remind lecture on ergodicity$\} =$

$$= \int_0^t dt_1 \int_0^{t_1} dt_2 \, B(t_2 - t_1) + \int_0^t dt_1 \int_{t_1}^t dt_2 \, B(t_2 - t_1) =$$

change order of integration and use $B(t_2 - t_1) = B(t_1 - t_2)$

$$= 2\int_0^t dt_1 \int_{t_1}^t dt_2 \, B(t_2 - t_1) = \{\tau = t_2 - t_1\} = 2\int_0^t dt_1 \int_0^{t-t_1} d\tau \, B(\tau) =$$

change order of integr.

$$= 2\int_0^t (t-\tau) B(\tau) d\tau$$, see details in ergodicity lecture

$$\boxed{\frac{dx}{dt} = \xi(t) \text{ stationary} \Rightarrow \langle x^2(t) \rangle = 2\int_0^t (t-\tau) B(\tau) d\tau}$$

$$dt \quad x(t) \text{ stationary} \Rightarrow \ldots = \int_0 \ldots$$

- Back to white noise $\xi(t)$

$$\langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t+\tau)\rangle \equiv B(\tau) = 2D\delta(\tau)$$

$$D = \int_0^\infty B(\tau)\,d\tau \Leftarrow \int_0^\infty \delta(\tau)\,d\tau = 1/2$$

1) $\langle x(t)\rangle = \left\langle \int_0^t \xi(t')\,dt' \right\rangle = \int_0^t \langle \xi(t')\rangle\,dt' = 0$

2) $\langle x^2(t)\rangle = 2\int_0^\infty (t-\tau)B(\tau)\,d\tau = 4D\int_0^\infty (t-\tau)\delta(\tau) =$

$$= 2Dt = Var[x(t)]$$

3) since $\xi(t)$ Gaussian $\Rightarrow x(t)$ Gaussian

From 1)-3) $\Rightarrow$ $p(x,t) = \frac{1}{\sqrt{2\pi Var[x(t)]}} \exp\left[-\frac{x^2}{2Var[x(t)]}\right] =$

$$= \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) \quad \text{Wiener (oBm)}$$

<u>Remark</u>. ACF once again

$$\langle x(t_1)x(t_2)\rangle = \left\langle \int_0^{t_1} dt_1'\,\xi(t_1') \int_0^{t_2} dt_2'\,\xi(t_2') \right\rangle =$$

$$= \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \langle \xi(t_1')\xi(t_2')\rangle = \{\text{Let } t_2 > t_1\} =$$

$$= \int_0^{t_1} dt_1' \int_0^{t_1} dt_2' \langle \xi(t_1')\xi(t_2')\rangle + \int_0^{t_1} dt_1' \int_{t_1}^{t_2} dt_2' \langle \xi(t_1')\xi(t_2')\rangle =$$

$$= 2D\int_0^{t_1} dt_1' \underbrace{\int_0^{t_1} dt_2'\,\delta(t_2'-t_1')}_{=1} + 2D\int_0^{t_1} dt_1' \underbrace{\int_{t_1}^{t_2} dt_2'\,\delta(t_2'-t_1')}_{\approx 0} =$$

$$= 2Dt_1, \quad t_1 < t_2$$

Similarly, $\langle x(t_1)x(t_2)\rangle = 2Dt_2, \quad t_2 < t_1$ $\Big\} \Rightarrow$

$\Rightarrow \langle x(t_1)x(t_2)\rangle = 2D\min\{t_1, t_2\}$ see above

<u>Remark 2</u>. Check independence of increments of oBm

Let $t_1 < t_2$ $\quad \langle x(t_1)[x(t_2) - x(t_1)] \rangle =$

$$= \langle x(t_1) x(t_2) \rangle - \langle x^2(t_1) \rangle = 0$$

The same for $t_1 < t_2 < t_3$

$\langle [x(t_2) - x(t_1)][x(t_3) - x(t_2)] \rangle = \{$open the breckets$\} =$

$$= 0$$

Remark 3 In physics lit-re often instead of $\frac{dx}{dt} = \xi(t)$, $\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t')$

$$\Updownarrow$$

$$\frac{dx}{dt} = \sqrt{2D}\,\eta(t), \quad \langle \eta(t)\eta(t') \rangle = \delta(t-t')$$

e) discrete time representation of white Gaussian noise

$$\frac{dW}{dt} = \xi(t), \quad t_{j-1} - t_j = \Delta t$$

$$W(t_{j+1}) - W(t_j) = \int_{t_j}^{t_j+\Delta t} \xi(t')\,dt' = ?$$

$$W(\Delta t) - W(0) = W(\Delta t) \underset{\uparrow}{=} (\Delta t)^{1/2} W(\underset{t=1}{1}) \qquad (**)$$

self-sim. $W(\lambda t) = \lambda^H W(t)$, $H = 1/2$

$$p_W(x,t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi D t}} \div \hat{p}_W(k,t) = \exp(-Dk^2 t)$$

$$W(1) \equiv \xi, \quad p_\xi(x) = \frac{e^{-x^2/4D}}{\sqrt{4\pi D}} \div \hat{p}_\xi(k) = e^{-Dk^2}$$

Let $\xi = \sqrt{2D}\,\eta \Rightarrow \{\hat{p}_\xi(k) = \hat{p}_\eta(\sqrt{2D}\,k)\} \Rightarrow$

$$\Rightarrow \hat{p}_\eta(k) = e^{-k^2/2} \div p_\eta(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

η : standard normal variable $\quad \eta \sim N(0,1)$, $\langle \eta \rangle = 0$, $\langle \eta^2 \rangle = 1$

Back to (**)

$$W(\Delta t) = (\Delta t)^{1/2} \xi = \sqrt{2D\Delta t}\,\eta$$

$$\boxed{\begin{aligned} &W(t_{j+1}) - W(t_j) = (2D\Delta t)^{1/2} \eta_j \\ &\eta_j \sim N(0,1) \end{aligned}} \Leftarrow$$

$$\Updownarrow$$

$$\boxed{\begin{aligned} &\frac{dW}{dt} = \xi(t) = \sqrt{2D}\,\eta(t) \\ &\langle \xi(t)\xi(t') \rangle = 2D\,\delta(t-t') \\ &\langle \eta(t)\eta(t') \rangle = \delta(t-t') \end{aligned}} \Leftarrow \cancel{\boxed{\begin{aligned} &dW = \xi(t)\,dt \\ &\Downarrow \\ &W(t_{j+1}) - W(t_j) = \\ &= \xi_j \Delta t \quad ? \end{aligned}}}$$

Remark 1. Math. definition of white Gaussian noise

$\xi(t)$ Gaussian stationary, such that

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \begin{cases} \sigma^2, & t = t' \\ 0, & t \neq t' \end{cases}$$