3.1. Definitions

$x(t)$ N realizations $x^{(j)}(t)$, $j=1,\dots,N$

(a) Ensemble averages

$$\langle x(t)\rangle = \lim_{N\to\infty}\frac{\sum_{j=1}^{N} x^{(j)}(t)}{N} = \int_{-\infty}^{\infty} x\,p(x,t)\,dx$$

$$\langle x^2(t)\rangle = \lim_{N\to\infty}\frac{\sum_{j=1}^{N}[x^{(j)}(t)]^2}{N} = \int_{-\infty}^{\infty} x^2 p(x,t)\,dx$$

ACF $$\langle x(t)x(t+\tau)\rangle = \lim_{N\to\infty}\frac{\sum_{j=1}^{N} x^{(j)}(t)\,x^{(j)}(t+\tau)}{N} =$$

$$= \int_{-\infty}^{\infty} x_1 x_2\, p_2(x_1,t;x_2,t+\tau)\,dx_1\,dx_2$$

(b) Time averages:

$$\overline{x^{(j)}(t)} = \frac{1}{T}\int_0^T x^{(j)}(t)\,dt$$

$$\overline{[x^{(j)}(t)]^2} = \frac{1}{T}\int_0^T [x^{(j)}(t)]^2\,dt$$

$$\overline{x^{(j)}(t)\,x^{(j)}(t+\tau)}^{\,T} = \frac{1}{T-\tau}\int_0^{T-\tau} x^{(j)}(t)\,x^{(j)}(t+\tau)\,dt$$

. . . .

Theory: ensemble average

Experiment: time average advantageous
number of realizations N is not big enough

Def. 1 Ergodic process:
all ensemble averages = corresponding time averages at $T\to\infty$

Def. 2. Weakly ergodic process (ergodicity in the wide sense)
(important in practice)

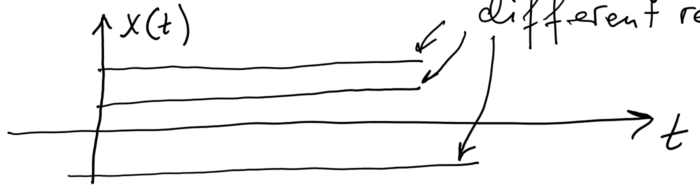
$$\left.\begin{aligned}\langle x(t)\rangle &= \overline{x^{(j)}(t)}, \quad T\to\infty\\ \langle x(t)x(t+\tau)\rangle &= \overline{x^{(j)}(t)\,x^{(j)}(t+\tau)}, \quad T\to\infty\end{aligned}\right\}\ \text{for any } j,\ j=1,\dots N$$

Theorem. Only stationary processes can be ergodic

Fact. "Most" of stationary processes are ergodic

Remark. Non-ergodic stationary process

different realizations



$x(t) = \xi$ r.v. $p(\xi)$

$$\langle x(t) \rangle = \int_{-\infty}^{\infty} \xi p(\xi) \, d\xi \quad \text{does not depend on time}$$

$$\langle x(t) x(t+\tau) \rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; x_1 x_2 \, p_2(x_1, t; x_2, t+\tau) =$$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; p(x_1) \, \delta(x_1 - x_2) \, x_1 x_2 \Rightarrow \int_{-\infty}^{\infty} dx_1 \; x_1^2 p(x_1) = \langle \xi^2 \rangle$$

However $\overline{x^{(j)}(t)} = \frac{1}{T} \int_0^T x^{(j)}(t) \, dt = x^{(j)} = \xi$ r.v.

3.2 Criteria of ergodicity

(a) role of Chebyshev inequality

$$x(t) = \langle x(t) \rangle + \xi(t), \qquad \langle \xi(t) \rangle = 0$$

$$\overline{x(t)} - \langle \overline{x(t)} \rangle = \overline{\xi(t)}$$

Reminder. Chebyshev inequality. X r.v.

$$\Pr\left(|X - \langle X \rangle| \geq \varepsilon \right) \leq \frac{\sigma^2}{\varepsilon^2}, \quad \sigma^2 = Var\{X\} = \langle (X - \langle X \rangle)^2 \rangle$$

Our case: $\overline{x(t)}$ is random variable

$$\Pr\left\{ \left| \overline{x(t)} - \langle \overline{x(t)} \rangle \right| \geq \varepsilon \right. \leq \frac{1}{\varepsilon^2} \underbrace{\langle \left(\overline{\xi(t)} \right)^2 \rangle}_{\sigma_T^2}$$

If $\sigma_T^2 \to 0$ at $T \to \infty \Rightarrow \overline{x(t)} \Rightarrow \langle \overline{x(t)} \rangle$

Consider $x(t)$ stationary process, $\langle x(t) \rangle = \langle x \rangle$

$$\langle \overline{x(t)} \rangle = \langle x \rangle$$

$$\boxed{\overline{\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) \, dt} = \langle x \rangle} \quad \text{if} \quad \lim_{T \to \infty} \sigma_T^2 = 0$$

(b) Time average in spectral representation

Any function $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega$

$$\overline{f(t)} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t')dt' = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} e^{-i\omega t'}dt'}\, d\omega \Rightarrow$$

$$= \{\underbrace{\quad} = \frac{1}{T}\,\frac{e^{-i\omega t'}}{-i\omega}\Big|_{t_0}^{t_0+T} = -\frac{1}{i\omega T}\left(e^{-i\omega(t_0+T)} - e^{-i\omega t_0}\right) =$$

$$= -\frac{e^{-i\omega t_0}}{i\omega T}\left(e^{-i\omega T} - 1\right) = -\frac{e^{-i\omega t_0}e^{-i\omega T/2}}{i\omega T}\underbrace{\left(e^{-\frac{i\omega T}{2}} - e^{\frac{i\omega T}{2}}\right)}_{-2i\sin\left(\frac{\omega T}{2}\right)} =$$

$$= \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} e^{-i\omega t_0} e^{-i\omega T/2} \}$$

$$\Rightarrow \boxed{\overline{f(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\, e^{-i\omega T/2} e^{-i\omega t_0}}$$

stronger damping of the high-frequency part of the spectrum

(c) spectral form of ergodicity criterion

$$\sigma_T^2 = \langle(\bar{\xi})^2\rangle = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} d\omega\, d\omega' \underbrace{\langle \xi(\omega)\xi(\omega')\rangle}_{2\pi G(\omega)\,\delta(\omega+\omega')} \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\,\frac{\sin\left(\frac{\omega' T}{2}\right)}{\frac{\omega' T}{2}} \times$$

previous lecture

$$\times\, e^{-\frac{i\omega T}{2} - \frac{i\omega' T}{2}} e^{-i\omega t_0 - i\omega' t_0}$$

$$= \int_{-\infty}^{\infty} G(\omega)\,\frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2}\,\frac{d\omega}{2\pi}$$

Example 1. Finite spectrum at $\omega = 0$, $0 < G(0) < \infty$

$$\sigma_T^2 \simeq \frac{G(0)}{T}\int_{-\infty}^{\infty}\frac{\sin^2(\omega T/2)}{(\omega T/2)^2}\,\frac{d(\omega T)}{2\pi} = \{ y = \frac{\omega T}{2} \} = \frac{G(0)}{T}\underbrace{\int_{-\infty}^{\infty}\frac{\sin^2 y}{y^2}\frac{dy}{\pi}}_{\text{convergent integral}} \sim$$

$$\sim \frac{G(0)}{T} \sim \frac{1}{T} \to 0, \quad T \to \infty$$

<u>Example 2.</u> $G(0) \to \infty$, flicker noise

$$G(\omega) \sim G_0 \left(\frac{\omega_0}{|\omega|}\right)^{\mu} \sim \frac{1}{|\omega|^{\mu}}, \quad 0 < \mu < 1$$

$$\sigma_T^2 \sim \frac{1}{T^{1-\mu}} \quad \text{slow convergence } \sigma_T^2 \to 0, \ T \to \infty$$

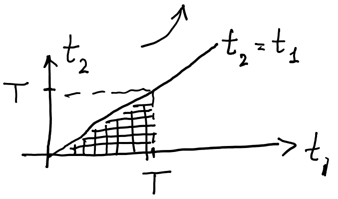
<u>Example 3.</u> $G(\omega) = G_0 \delta(\omega)$ non-ergodic stationary process, see example above

$$\sigma_T^2 = G_0 = \text{const}$$

(d) ergodicity criterion in time domain (Slutsky theorem)

$$\sigma_T^2 = \langle (\bar{\xi})^2 \rangle = \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \underbrace{\langle \xi(t_1)\xi(t_2) \rangle}_{B(\tau);\ \tau = t_2 - t_1} =$$

$$= \frac{1}{T^2} \int_0^T dt_1 \left[\int_0^{t_1} dt_2\, B(t_2 - t_1) + \int_{t_1}^T dt_2\, B(t_2 - t_1) \right] =$$

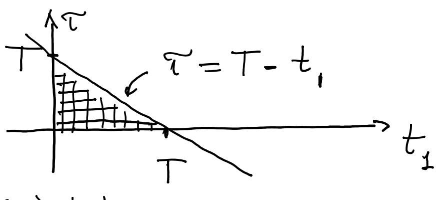


$$= \frac{1}{T^2} \int_0^T dt_2 \int_{t_2}^T dt_1\, B(t_2 - t_1) + \frac{1}{T^2} \int_0^T dt_1 \int_{t_1}^T dt_2\, B(t_2 - t_1) = \{ B(\tau) = B(-\tau) \}$$

$t_1 \leftrightarrow t_2$

$$= \frac{2}{T^2} \int_0^T dt_1 \int_{t_1}^T dt_2\, B(t_2 - t_1) = \{ \tau = t_2 - t_1 \} =$$

$$= \frac{2}{T^2} \int_0^T dt_1 \int_0^{T - t_1} d\tau\, B(\tau) =$$



$$= \frac{2}{T^2} \int_0^T d\tau\, B(\tau) \int_0^{T-\tau} dt_1 = \underbrace{\frac{2}{T^2} \int_0^T (T - \tau) B(\tau)\, d\tau = \sigma_T^2}$$

$$\boxed{\lim_{T \to \infty} \frac{1}{T} \int_0^T B(\tau)\, d\tau = 0}$$

Slutsky Theorem (suff. and necessary cond of ergodicity)

Example 1. Fractional Gaussian noise

$$B(\tau) \sim \frac{1}{\tau^{\mu}}, \quad \tau \to \infty, \quad 0 < \mu < 1$$

$$\frac{1}{T}\int_0^T B(\tau)\,d\tau \sim \frac{1}{T^{1-\mu}} \to 0$$

Example 2. $B(\tau) = \text{const} \cdot \cos\omega\tau$ (previous lecture)

$$\frac{1}{T}\int_0^T B(\tau)\,d\tau \sim \frac{\sin\omega T}{\omega T} \to 0$$

Remark We consider only ergodicity in the mean sense

$$x(t) = \langle x \rangle + \xi(t)$$

(e) 1-point generalization $f[x(t)]$

$$\overline{f_T[x(t)]}^{\,T} = \frac{1}{T}\int_0^T f[x(t)]\,dt$$

Example $\overline{x^2(t)}^{\,T} \to \langle x^2(t) \rangle$, $T \to \infty$

- Construct analogue of $B(t, t') = \langle \xi(t)\xi(t') \rangle$

$$\psi_f(t, t') = \iint_{-\infty}^{\infty} f(x) f(x') \left[p_2(x, t; x', t') - p(x, t) p(x', t') \right] dx\,dx'$$

- Chebyshev inequality

$$\Pr\left\{ \left| \overline{f_T(x)} - \langle \overline{f_T(x)} \rangle \right| \geq \varepsilon \right\} \leq \frac{Var\left[\overline{f_T(x)}\right]}{\varepsilon^2}$$

- For stationary processes

$$\psi_f(t, t') = \psi_f(\tau = t' - t)$$

$$Var\left[\overline{f_T(x)}\right] = \frac{1}{T^2}\int_0^T dt_1 \int_0^T dt_2\, \psi_f(t_1, t_2) = \left\{ \begin{matrix}\text{the same}\\ \text{procedure}\\ \text{as above}\end{matrix} \right\} \Rightarrow$$

$$\Rightarrow \boxed{\lim_{T \to \infty} \frac{1}{T}\int_0^T \psi_f(\tau)\,d\tau = 0} \quad \text{Slutsky}$$

<u>Example</u> $f(x) = x^2(t)$

$$\psi_f(\tau) = \left\langle \left(x^2(t) - \langle x^2\rangle\right)\left(x^2(t+\tau) - \langle x^2\rangle\right)\right\rangle$$

(f) 2-point for stationary process

Let $\langle x\rangle = 0$

Q. $\overline{x(t)x(t+\tau)}^{T} = \frac{1}{T-\tau}\int_0^{T-\tau} x(t)\,x(t+\tau)\,d\tau \rightarrow$

$\rightarrow \langle x(t)\,x(t+\tau)\rangle$?

A. $\mathrm{Var}\left\{\overline{x(t)\,x(t+\tau)}^{T}\right\} \xrightarrow[T\to\infty]{} 0$

Introduce ACF of $\eta_\tau(t) = \xi(t)\xi(t+\tau)$

$$\text{ACF} \equiv \psi_\eta(\sigma) = \langle \eta_\tau(t)\eta_\tau(t+\sigma)\rangle - \langle\eta_\tau(t)\rangle\langle\eta_\tau(t+\sigma)\rangle =$$

$$= \langle \xi(t)\xi(t+\tau)\xi(t+\sigma)\xi(t+\sigma+\tau)\rangle -$$

$$- \langle \xi(t)\xi(t+\tau)\rangle \underbrace{\langle \xi(t+\sigma)\xi(t+\tau+\sigma)\rangle}_{\langle \xi(t)\xi(t+\tau)\rangle}$$

$$\boxed{\lim_{T\to\infty}\frac{1}{T}\int_0^T \psi_\eta(\sigma)\,d\sigma = 0}$$ Slutsky

<u>Remark 1</u> As a rule: $\langle \eta_\tau(t)\eta_\tau(t+\tau)\rangle \xrightarrow[\sigma\to\infty]{} \langle\eta_\tau(t)\rangle\langle\eta_\tau(t+\tau)\rangle \Rightarrow$

$\Rightarrow$ OK

<u>Remark 2</u>. Example of nonstationary process that is ergodic in the mean

$x(t) = x_0 + a\cos\omega t$, a r.v., $\langle a\rangle = 0$, $x_0 = \langle x(t)\rangle = \text{const}$

Nonstationary: $\langle (x-x_0)^2\rangle = \langle a^2\rangle\cos^2(\omega t) \Rightarrow \mathrm{Var}[x(t)]$ is time-dependent $\Rightarrow$ $x(t)$ non-stationary

However,

$\overline{x(t)} = \frac{1}{T}\int_0^T \left(x_0 + a\cos(\omega t)\right)dt = x_0 = \langle x(t)\rangle$ $x(t)$ is mean ergodic

But $f(x) = x^2(t) = y(t)$

$$y(t) = x_0^2 + 2x_0 a\cos\omega t + a^2\cos^2\omega t$$

$$\left.\langle y(t)\rangle = x_0^2 + \langle a^2\rangle\cos^2\omega t\right\} \Rightarrow \overline{y(t)} \neq \langle y(t)\rangle$$

$$\left.\begin{aligned} \langle y(t)\rangle &= x_0^2 + \langle a^2\rangle \cos^2\omega t \\ \overline{y(t)} &= x_0^2 + \frac{a^2}{2} \end{aligned}\right\} \Rightarrow \overline{y(t)} \neq \langle y(t)\rangle$$

II.4. Important classes of non-stationary processes

4.1. Nonstationary process with independent increments

Consider $x(t)$: sequence of time instants $t_1 < t_2 < \ldots < t_n$

(a) Definition :

$$x(t_1),\ \underbrace{x(t_2) - x(t_1)}_{\text{increment}},\ x(t_3) - x(t_2), \ldots, x(t_n) - x(t_{n-1})$$

are statistically independent

$$x(t_n) = x(t_1) + \sum_{j=2}^{n} \left[x(t_j) - x(t_{j-1}) \right]$$

Remind : the stoch process is defined if we know all n-point probability densities for all n and all time instants

n-point CF $\hat{p}_n(k_1, t_1; \ldots k_n, t_n) = \langle \exp\left[i\left(k_1 x(t_1) + \ldots + k_n x(t_n)\right)\right]\rangle =$

$$= \langle \exp\left[i(k_1 + \ldots + k_n)x(t_1) + i(k_2 + \ldots + k_n)\left(x(t_2) - x(t_1)\right) + \right.$$

$$\left. + i(k_3 + \ldots + k_n)\left(x(t_3) - x(t_2)\right) + \ldots + ik_n\left(x(t_n) - x(t_{n-1})\right)\right]\rangle =$$

$$= \langle \exp\left[i(k_1 + \ldots + k_n)x(t_1)\right]\rangle \langle \exp\left[i(k_2 + \ldots + k_n)\left(x(t_2) - x(t_1)\right)\right]\rangle$$

$$\times \ldots \times \langle \exp\left[ik_n\left(x(t_n) - x(t_{n-1})\right)\right]\rangle =$$

$$= \hat{p}(k_1 + \ldots + k_n; t_1)\, \hat{p}(k_2 + \ldots + k_n; t_1, t_2) \ldots \times \hat{p}(k_n; t_{n-1}, t_n)$$

$\hat{p}(k, t) = \langle e^{ikx(t)}\rangle$ CF of $x(t)$

$\hat{p}(k, t_1, t_2) = \langle e^{ik(x(t_2) - x(t_1))}\rangle$ CF of increments of $x(t)$

$$\hat{p}_2(k_1, t_1; k_2, t_2) = \langle e^{ik_1 x_1 + ik_2 x_2}\rangle$$

$$\hat{p}(k, t_1, t_2) = \hat{p}_2(-k, t_1; k, t_2)$$

(b) Gaussian process with independent increments

$x(t)$ Gaussian

$x(t_2) - x(t_1)$ Gaussian

<u>Remind</u>. Gaussian r.v. $\hat{p}_X(k) = \exp\left[i\mu k - \frac{\sigma^2 k^2}{2}\right]$

$$p_X(x) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\mu = \langle X \rangle$$

$$\sigma^2 \equiv Var\{X\} = \langle (X - \langle X \rangle)^2 \rangle$$

$$\langle X^n \rangle = \frac{1}{i^n} \left. \frac{d^n \hat{p}(k)}{dk^n} \right|_{k=0}$$

$$\hat{p}(k,t) = \exp\left[i\mu(t)k - \frac{1}{2}\sigma^2(t)k^2\right]$$

$$\hat{p}(k,t_1,t_2) = \langle e^{ik(x(t_2)-x(t_1))} \rangle =$$

$$= \exp\left[i\mu_\Delta(t_1,t_2)k - \frac{1}{2}\sigma_\Delta^2(t_1,t_2)k^2\right], \text{ where}$$

$$\mu(t) = \langle x(t)$$

$$\sigma^2(t) = Var\{x(t)\} = \langle (x(t) - \langle x(t) \rangle)^2 \rangle = \langle x^2(t) \rangle - \langle x(t) \rangle^2$$

$$\mu_\Delta(t_1,t_2) = \langle (x(t_2) - x(t_1) \rangle$$

$$\sigma_\Delta(t_1,t_2) = Var\{(x(t_2) - x(t_1))\}$$

$$x(t_2) = \underbrace{x(t_1) + (x(t_2) - x(t_1))}_{\text{statistically independent}} \Rightarrow \sigma(t_2) = \sigma(t_1) + \sigma_\Delta(t_1,t_2)$$

$$\Rightarrow \quad \mu_\Delta(t_1,t_2) = \langle (x(t_2) - x(t_1)) \rangle = \mu(t_2) - \mu(t_1)$$

$$\sigma_\Delta(t_1,t_2) = \sigma(t_2) - \sigma(t_1)$$

PDFs $\quad p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}}\exp\left[-\frac{(x-\mu(t))^2}{2\sigma^2(t)}\right]$

$$p(x_2 - x_1; t_1, t_2) = \frac{1}{\sqrt{2\pi(\sigma^2(t_2) - \sigma^2(t_1))}}\exp\left[-\frac{(x_2 - x_1 - \mu(t_2) + \mu(t_1))^2}{2(\sigma^2(t_2) - \sigma^2(t_1))}\right]$$

Example 1. Wiener process (ordinary Brownian motion)

$$\mu(t) = 0, \quad \sigma^2(t) = ct, \quad c = \text{const}$$

Applications: everywhere

Example 2 Scaled Brownian motion

$$\mu(t) = 0, \quad \sigma^2(t) = ct^\alpha, \quad \alpha > 0, \quad c = \text{const}$$

Applications: granular gases, hydrology, finance...

Remarks. 1) adding $\mu(t) \neq 0$ is trivial

2) terminology Wiener process $\sigma^2(t) = t$, $c = 1$