10.1. Definition, examples, properties...

Def. Probability law infinitely divisible, if at any $n = 0, 1, 2, \ldots$

$$\hat{p}(k) = [\hat{p}_n(k)]^n,$$

where $\hat{p}_n(k)$ is some other CF

Example 1 Gaussian law (normal law)

CF $\hat{p}(k) = \exp\left[i\mu k - \frac{\sigma^2 k^2}{2}\right]$

PDF $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

$\langle x \rangle = \mu$, $Var[x] = \sigma^2$

$$\hat{p}_n(k) = [\hat{p}(k)]^{1/n} = \exp\left[i\frac{\mu}{n}k - \frac{\sigma^2 k^2}{2n}\right],$$

Gaussian with the mean $= \frac{\mu}{n}$, variance $= \frac{\sigma^2}{n}$

Example 2 Poisson law

r.v. ξ can get only values $am + b$,
a, b real, $m = 0, 1, 2, \ldots$

$P(\xi = am + b) = \frac{e^{-\lambda}\lambda^m}{m!}$ Probability that ξ can take a value $\xi = am + b$

Normalization: $\sum_{m=0}^{\infty} P(\xi = am + b) = 1$

$$\text{CF } \hat{p}_\xi(k) = \langle e^{ik\xi} \rangle = \sum_{m=0}^{\infty} e^{ik(am+b)} P(\xi = am + b) =$$

$$= \sum_{m=0}^{\infty} e^{iamk + ibk} \frac{e^{-\lambda}\lambda^m}{m!} = e^{-\lambda + ibk} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{iamk} =$$

$$= e^{-\lambda + ibk} \sum_{m=0}^{\infty} \frac{(e^{iak}\lambda)^m}{m!} = e^{-\lambda + ibk} e^{\lambda e^{iak}} \Rightarrow$$

$$\Rightarrow \boxed{\hat{p}_\xi(k) = \exp\{ibk + \lambda(e^{iak} - 1)\}}$$

Normalization $\hat{p}_\xi(0) = 1$

$$\left[\hat{p}_n(k)\right] = \left[\hat{p}_\xi(k)\right]^{1/n} = \exp\left\{i\frac{b}{n}k + \frac{\lambda}{n}\left(e^{iak}-1\right)\right\}$$

CF of the Poisson law with parameters $a, \frac{b}{n}, \frac{\lambda}{n}$

Exercise. $\langle \xi \rangle = ?$ $Var\{\xi\} = ?$

Property 1. Distribution function of the sum of independent r.v. having inf. divisible distributions is also infinitely divisible.

Proof Consider 2 independent r.v. ξ and η

$$\hat{p}_\xi(k) = \left[\hat{p}_n(k)\right]^n, \quad \hat{p}_\eta(k) = \left[\hat{f}_n(k)\right]^n$$

The sum of ind. r.v.

$$\hat{p}_{\xi+\eta}(k) = \hat{p}_\xi(k)\hat{f}_\eta(k) = \left[\hat{p}_n(k)\hat{f}_n(k)\right]^n \Rightarrow$$

$\hat{p}_{\xi+\eta}(k)$ is CF of inf divisible law

10.2 The canonical representation of infinitely divisible laws

Assumption: finite variance (with infinite variance later on)

Theorem. For a distribution function with finite variance to be infinitely divisible, it is necessary and sufficient that

$$\boxed{\ln\hat{p}(k) = i\mu t + \int_{-\infty}^{\infty}\left\{e^{ikx} - 1 - ikx\right\}\frac{1}{x^2}G(x)}$$ Kolmogorov

where μ is real constant, and G is a non-decreasing function of bounded variation

<u>Remark.</u> Real function f is of bounded variation (BV) if and only if it can be written as the difference $f = f_1 - f_2$ of two non-decreasing functions f_1 and f_2 (Jorden decomposition of a measure)

Proof of necessity

Assume that $p(x)$ is the PDF of inf divisible law.

$$\hat{p}(k) = [\hat{p}_n(k)]^n$$

$\uparrow$ some CF

$$\ln \hat{p}(k) = n \ln \hat{p}_n(k) = n \ln[1 + (\hat{p}_n(k) - 1)] \quad (1)$$

Since $\hat{p}_n(k) = [\hat{p}(k)]^{1/n}$, then $\hat{p}_n(k) \to 1$, $n \to \infty$

$\Downarrow$

$\hat{p}_n(k) - 1$ is small at $n \to \infty$ $\swarrow$ $\ln(1+x) \simeq x$

$$\ln[1 + (\hat{p}_n(k) - 1)] \underset{n \to \infty}{\simeq} (\hat{p}_n(k) - 1) \quad (2)$$

Plug (2) into (1)

$$\ln \hat{p}(k) \underset{n \to \infty}{=} n(\hat{p}_n(k) - 1) = n \int_{-\infty}^{\infty} (e^{ikx} - 1) p_n(x)\, dx. \quad (3)$$

where $\hat{p}_n(k) = \int_{-\infty}^{\infty} e^{ikx} p_n(x)\, dx$

Inset. Expectation value (the mean)

Inf. divisible $\hat{p}(k) = [p_n(k)]^n$

Recall that $\hat{p}(k) = \int e^{ikx} p(x)\, dx \Rightarrow$

$$\Rightarrow \frac{1}{i} \frac{d\hat{p}(k)}{dk} = \int_{-\infty}^{\infty} x e^{ikx} p(x)\, dx$$

$$\frac{1}{i} \frac{d\hat{p}(k)}{dk}\bigg|_{k=0} = \int_{-\infty}^{\infty} x p(x)\, dx = \langle x \rangle$$

From inf divisibility follows that

$$\frac{1}{i}\frac{d\hat{p}(k)}{dk} = \frac{1}{i}\, n\,[p_n(k)]^{n-1}\,\frac{d\hat{p}_n(k)}{dk}$$

$$\frac{1}{i}\frac{d\hat{p}(k)}{dk}\Big|_{k=0} = \frac{1}{i}\, n\,\underbrace{[\hat{p}_n(k=0)]^{n-1}}_{=1 \text{ normalization}}\,\frac{d\hat{p}_n(k)}{dk}\Big|_{k=0}$$

$$\int_{-\infty}^{\infty} x p(x)\,dx = n\int_{-\infty}^{\infty} x\, p_n(x)\,dx = \mu \qquad (4)$$

Plug (4) into (3)

$$\underline{\ln\hat{p}(k)} = n\int_{-\infty}^{\infty}(e^{ikx}-1)\,p_n(x)\,dx + i\mu k - ikn\int_{-\infty}^{\infty} x\, p_n(x)\,dx =$$

$$= \underbrace{i\mu k + n\int_{-\infty}^{\infty}(e^{ikx}-1-ikx)\,p_n(x)\,dx} \qquad (5)$$

Introduce $\quad G_n(x) = n\int_{-\infty}^{x}\underbrace{u^2 p_n(u)}_{\text{non-negative}}\,du \qquad (6)$

(i) $G_n(x)$ non-decreasing $\quad \Rightarrow dG_n(x) = nx^2 p_n(x)\,dx$

(ii) $G_n(-\infty) = 0$

(iii) $G_n(x)$ is uniformly bounded

<u>Proof</u>. $\quad G_n(+\infty) = n\int_{-\infty}^{\infty} u^2 p_n(u)\,du =$

$$= n\left[u^2 p_n(u)\,du - \left(\int_{-\infty}^{\infty} u\, p_n(u)\,du\right)^2\right] +$$

$$+ n\left(\int_{-\infty}^{\infty} u\, p_n(u)\,du\right)^2 \underset{(4)}{=} \underbrace{\sigma^2 + \frac{1}{n}\mu^2} \qquad (*)$$

where σ^2 is variance of the law $p(x)$.

Plug (6) into (5). In new notations

$$\ln\hat{p}(k) = i\mu k + \lim_{n\to\infty}\int_{-\infty}^{\infty}\left(e^{ikx}-1-ikx\right)\frac{1}{x^2}\,dG_n(x). \qquad (7)$$

$$\ln \hat{p}(k) = i\mu k + \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2} dG_n(x) . \quad (7)$$

"Math" part of the proof: existence of the $\lim\limits_{n\to\infty}$ stems from (i)-(iii) properties of G_n, see (*)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2} dG_n(x) = \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2} dG(x) . \quad (8)$$

$$\Downarrow$$

If the probability law is infinitely divisible $\Rightarrow$
$\Rightarrow$ its characteristic function $\hat{p}(k)$ such that

$$\boxed{\ln \hat{p}(k) = i\mu k + \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2} dG(x)} \quad (9)$$

Proof of sufficiency

If (9) holds $\Rightarrow$ $\hat{p}(k)$ is CF of infinitely divisible law

Step 1 For any ε $(0 < \varepsilon < 1)$ the integral

$$\int_{\varepsilon}^{1/\varepsilon} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2} dG(x) \quad (10)$$

is the limit of the sum

$$\sum_{s=1}^{n} \left(e^{ik\bar{x}_s} - 1 - ik\bar{x}_s\right) \frac{1}{\bar{x}_s^{\,2}} \left(G(x_{s+1}) - G(x_s)\right), \quad (11)$$

$x_1 = \varepsilon$, $x_{n+1} = \frac{1}{\varepsilon}$, $x_s \le \bar{x}_s \le x_{s+1}$, and $\max(x_{s+1} - x_s) \to 0$

Reminder. Poisson law $P(\xi = am + b) = \dfrac{e^{-\lambda}\lambda^m}{m!}$ $\Rightarrow$

$\Rightarrow \hat{p}_\xi(k) = \exp\left\{\lambda\left(e^{iak} - 1\right) + ibk\right\}$

Each term of the sum (11) is the logarithm of the CF of some Poisson law

Poisson law is inf divisible $\underset{\substack{\uparrow \\ \text{Property 1}}}{\Longrightarrow}$ the sum inf divisible laws $\Rightarrow$

$\Rightarrow$ the integral (10) is the logarithm of some inf. divisible law

<u>Step 2</u> Take the limit $\varepsilon \to 0 \Rightarrow$

$$\Rightarrow \int_{x>0} \left(e^{itx} - 1 - itx\right) \frac{1}{x^2}\, dG(x) \qquad (12a)$$

is the log of CF of inf div law AND

$$\int_{x<0} \left(e^{itx} - 1 - itx\right) \frac{1}{x^2}\, dG(x) \qquad (12b)$$

is the log of the CF of inf div law

Go to (9) and subtract (12a) and (12b):

$$\ln \hat{p}(k) - (12a) - (12b) = i\mu k + \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2}\, dG(x) -$$

$$- (12a) - (12b) =$$

$$= i\mu k + \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} \left(\underbrace{e^{ikx}}_{= 1 + ikx - \frac{1}{2}k^2x^2} - 1 - ikx\right) \frac{1}{x^2}\, dG(x) =$$

$$= i\mu k - \frac{k^2}{2}\left(G(0^+) - G(0^-)\right) \Rightarrow$$

Recall Gaussian PDF $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \div$

$$CF = \exp\left[i\mu k - \frac{\sigma^2 k^2}{2}\right]$$

$\Rightarrow$ ln CF of the Gaussian law with mean μ and the variance

$$\sigma^2 = \frac{1}{2}\left(G(0^+) - G(0^-)\right)$$

Gaussian law is inf divisible !

<u>Conclusion</u> $\hat{p}(k)$ is the CF of inf. divisible law (property 1).

$$\ln \hat{p}(k) = i\mu k + \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2}\, dG(x) =$$

$$= i\mu k - \frac{\sigma^2 k^2}{2} + \int_{0^+}^{\infty} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2}\, dG(x) +$$

$$+ \int_{-\infty}^{0^-} \left(e^{ikx} - 1 - ikx\right) \frac{1}{x^2}\, dG(x). \qquad (13)$$

where $G(x)$ is a non-decreasing non-negative function

and $\sigma^2 = \frac{1}{2}\left(G(0^+) - G(0^-)\right)$

Remark 1. Gaussian law and Poisson laws are the basic elements that comprise any infinitely divisible law

Remark 2 Probabilistic meaning.

Differentiate (13) twice.

$$\frac{d^2}{dk^2} \ln \hat{p}(k) = -\int_{-\infty}^{\infty} e^{ikx}\, dG(x) \qquad (14)$$

$$\Downarrow$$

$G(x)$ is uniquely determined by the function $\hat{p}(k)$

Reminder Cumulants (1st lecture)

$$\varkappa_n = \frac{1}{i^n} \frac{d^n}{dk^n} \left(\ln \hat{p}(k)\right)\Big|_{k=0}$$

$$\mathrm{Var} = \mu_2 - \mu_1^2 = \varkappa_2$$

From (14) $\mathrm{Var}\{\xi\} = -\left[\frac{d^2}{dk^2} \ln \hat{p}(k)\right]_{k=0} = \int_{-\infty}^{\infty} dG(x) = G(+\infty) \quad (15)$

Also, remembering (4), $\mu = \langle \xi \rangle$

Example 1. Canonical representation of the normal law

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \div \exp\left[i\mu k - \frac{\sigma^2 k^2}{2}\right]$$

Check

$$\int_{-\infty}^{\infty} \left\{ e^{ikx} - 1 - ikx \right\} \frac{1}{x^2} \, dG(x) = - \frac{\sigma^2 k^2}{2}$$

if $G(x) = \begin{cases} 0, & x < 0 \\ \sigma^2, & x > 0 \end{cases}$

<u>Example 2</u> Poisson law

$$\hat{p}(k) = \exp\left\{ \lambda \left(e^{iak} - 1 \right) + ibk \right\}$$

$G(x)$ has a single jump at the point a,

$$G(x) = \begin{cases} 0, & x < a \\ a^2 \lambda, & x > a \end{cases} \quad , \quad \mu = \langle \xi \rangle = b + a\lambda$$

Another example of infinitely divisible distributions

(i) Cauchy

(ii) Gamma

(iii) Laplace

<u>Remark</u>. We considered infinitely divisible distributions <u>with finite variance</u>

For infinite variance in the Chapter "Stochastic Processes"

Inf divisibility $\Rightarrow$ Lévy processes