WiSe 2024-25. StochPro I.
Lectures 1-2.

Plan. StochPro-Part I.

I. Random Variables

1. Basic definitions and properties
2. Central Limit Theorem
3. Alpa-stable probability laws
4. Generalized Central Limit Theorem

II. Stochastic Processes

1.

- van Kampen
- Gardiner
- Capasso, Bakstein Stoch. Processes ...
- Baschnagel Stoch. Processes from Phys to Finance

- Paulos Nimtz
- Maximilian Steuer
- Alek Schiffer

Lectures 1-2 WiSe 24-25

Stochastic Processes and Statistical Methods

I. STOCHASTIC VARIABLES.

CONTENT

ABBREVIATIONS

PDF = probability density function
CF = characteristic function
CDF = cumulative distribution function
r.v. = random variable
i.r.v. = independent random variables
iid r.v. = independent identically distributed random variables
CLT = Central Limit Theorem
GCLT = Generalized Central Limit Theorem
lhs = left hand side (of an equation)
rhs = right hand side (of an equation)

NOTATIONS

$\div$ Fourier (Laplace) transform pair

$X \stackrel{d}{=} Y$ = equality in distribution

NOTATIONS

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r.v. = random variable
i.r.v. = independent random variables
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CLT = Central Limit Theorem

1. Basic definitions

1.1. Stochastic variable (random variable, random number)

Definition. A "*random number*" or "*stochastic variable*" X is an object defined by

- set of possible values Ω (called "*range*", "*set of states*", "*sample space*" or "*phase space*");
- probability distribution over this set;

1.2. Set Ω can be

- discrete (number of particles in a reacting mixture);
- continuous (velocity component of Brownian particle, $-\infty < v_x < \infty$; kinetic energy of Brownian particle, $0 < E < \infty$);
- continuous and discrete (energy of electrons in presence of binding states);
- multidimensional; Notation: $\vec{X}$.

1.3. Probability density function (PDF) (probability distribution)

Notation. $p_X(x)$ or simply $p(x)$.

Meaning. $p(x)dx = \Pr\{x \le X < x + dx\}$

Properties.

(i) $p(x) \ge 0$

(ii) $\int_\Omega p(x)dx = 1$

1.4. Cumulative distribution function (CDF).

$$P(x) := \Pr\{X \le x\} = \int_{-\infty}^{x^+} p(x')dx' .$$

Remark: "$-\infty$" at a lower bound with no loss of generality. Suppose $a \le X \le b$. Then

$$p(x) \to \bar{p}(x) = \begin{cases} p(x) & , \quad a \le x \le b \\ 0 & , \quad otherwise \end{cases} .$$

2. Averages

2.1. Average (expectation value)

r.v. X, function $g(X)$ of the random variable X

$$\langle g(X) \rangle := \int_\Omega g(x)p(x)dx \qquad \text{Math notation: } \mathbf{E}[g(X)] \text{ (Expectation value)}$$

2.2. Moments

$g(X) = X^n$, $n = 0.1, 2,$

$$\langle X^n \rangle := \int_{\Omega} x^n p(x)dx \equiv \mu_n \quad .$$

Add central moments $\overline{\mu_n}$!

$\rightarrow$ first moment (mean, average) $\langle X \rangle$, $\mathbf{E}[X]$

$\rightarrow$ second moment $\langle X^2 \rangle$, $\mathbf{E}\left[X^2\right]$

$\rightarrow$ variance (dispersion) $\sigma^2 := \left\langle \left(X - \langle X \rangle\right)^2 \right\rangle = Var[X] = ... = \mu_2 - \mu_1^2$,

$\rightarrow$ standard deviation σ .

The second moment characterizes intensity of stochastic process, variance characterizes intensity of fluctuations. $\mapsto$ next page: Digression

2.3. Chebyshev inequality (*important in data analysis*)

$\boxed{\Pr\left(|X - \mu_1| \geq r\sigma\right) = ?}$, r positive constant .

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu_1)^2 p(x)dx = \left(\int_{-\infty}^{\mu_1 - r\sigma} + \int_{\mu_1 - r\sigma}^{\mu_1 + r\sigma} + \int_{\mu_1 + r\sigma}^{\infty} \right) (x - \mu_1)^2 p(x)dx \geq$$

$$\geq \left(\int_{-\infty}^{\mu_1 - r\sigma} + \int_{\mu_1 + r\sigma}^{\infty} \right) (x - \mu_1)^2 p(x)dx \ .\sim$$

In the last two integrals $(x - \mu_1)^2 \geq r^2\sigma^2$, thus if we replace $(x - \mu_1)^2 \rightarrow r^2\sigma^2$, then

$$\sigma^2 \geq r^2\sigma^2 \left(\int_{-\infty}^{\mu_1 - r\sigma} + \int_{\mu_1 + r\sigma}^{\infty} \right) p(x)dx = r^2\sigma^2 \Pr\left\{|x - \mu_1| \geq r\sigma\right\} \quad ,$$

therefore

$$\boxed{\Pr\left\{|X - \mu_1| \geq r\sigma\right\} \leq \frac{1}{r^2}} \quad \text{(Chebyshev)}$$

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<u>Consequences.</u>

- $r < 1$ trivial;
- $r = \sqrt{2}$ max 50% of the total probability is beyond $\sqrt{2}$ times standard deviation;
- $r = 2$ max 25% beyond two standard deviations;
- $r = 3$ max 11.(1)% beyond 3 standard deviations;

<u>Equivalent form</u>. $r\sigma \equiv \varepsilon$

$$\boxed{\Pr\left\{|X - \mu_1| \geq \varepsilon\right\} \leq \frac{\sigma^2}{\varepsilon^2}} \qquad (1.2)$$

any distribution with finite 2nd mom.

Remark.

(i) *Three-sigma rule of thumb for Gaussian distribution*: $r = 3 \Rightarrow \sim 0.9973$.

Digression. R.v. *X* discrete set $\Omega = \{x_1, x_2, \ldots, x_N\}$

$p_X(x) = ?$

$$p_X(x) = p_1\delta(x - x_1) + p_2\delta(x - x_2) + \ldots + p_N\delta(x - x_N) = \sum_{i=1}^{N} p_i\delta(x - x_i)$$

- Normalization

$$\int_{-\infty}^{\infty} p_X(x)dx = 1 \quad \Rightarrow \quad \sum_{i=1}^{N} p_i = 1$$

- Expectation value

$$\langle X \rangle = \int_{-\infty}^{\infty} x\, p_X(x)dx = \sum_{i=1}^{N} x_i p_i$$

- M trials (probabilistic experiment), M >> 1.

X = x_1 in M_1 trials,
X = x_2 in M_2 trials
.... M1 + M2 + … + MN = M
X = x_N in M_N

•• experimental probabilities

$$p_{1\,\text{exp}} \approx \frac{M_1}{M}, \quad p_{2\,\text{exp}} \approx \frac{M_2}{M}, \quad \ldots, \quad p_{N\,\text{exp}} \approx \frac{M_N}{M}$$

•• experimental mean

$$\langle X \rangle_{\text{exp}} = \frac{1}{M}\{x_1 M_1 + \ldots + x_N M_N\} = \frac{1}{M}\sum_{i=1}^{N} x_i M_i$$

•• at $M \to \infty \quad \Rightarrow \quad p_{i\,\text{exp}} \to p_i, \quad \langle X \rangle_{\text{exp}} \to \langle X \rangle$

Back to Chebyshev inequality

3. Characteristic function (CF)

Definition.

$$\hat{p}(k) := \int_{-\infty}^{\infty} e^{ikx} p(x)dx \equiv \left\langle e^{ikX} \right\rangle \quad x, k \in \mathbf{R}$$

Fourier transform pair (symbol ÷):

$$\hat{p}(k) \div p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{p}(k)dk \quad .$$

Note prefactor

Properties.

1) $\hat{p}(0) = 1$.

2) $|\hat{p}(k)| \le \int_{-\infty}^{\infty} \left| e^{ikx} \right| p(x)dx = 1$.

3) $\hat{p}(-k) = \hat{p}^*(k)$.

4) CF as the moment generating function

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)dx = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(ikx)^n}{n!} p(x)dx = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \int_{-\infty}^{\infty} x^n p(x)dx = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n =$$

$$= 1 + ik\mu_1 - \frac{k^2}{2}\mu_2 + \ldots \quad ,$$

thus

$$\mu_1 = \frac{1}{i} \frac{d\hat{p}(k)}{dk}\bigg|_{k=0} \quad ,$$

$$\mu_2 = \frac{1}{i^2} \frac{d^2\hat{p}(k)}{dk^2}\bigg|_{k=0} = -\frac{d^2\hat{p}(k)}{dk^2}\bigg|_{k=0} \quad ,$$

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and so on. The general formula reads

$$\boxed{\mu_n = \frac{1}{i^n} \frac{d^n\hat{p}(k)}{dk^n}\bigg|_{k=0}} \quad , \tag{1.3}$$

5) CF as generating function for cumulants

Use

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad , \quad x^2 < 1, x = -1 \quad .$$

Let $1 - x = \hat{p}(k)$ (note property 2 of the CF). Then

$$\hat{p}(k) = \exp\left[-\sum_{n=1}^{\infty}\frac{(1-\hat{p}(k))^n}{n}\right] .$$

On the other hand (see property 4 of the CF)

$$\hat{p}(k) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mu_n ,$$

or

$$\hat{p}(k) - 1 = \sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\mu_n ,$$

Therefore

$$\hat{p}(k) = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{n=1}^{\infty}\frac{(ik)^m}{m!}\mu_m\right)^n\right] ,$$

or

$$\boxed{\ln \hat{p}(k) = \sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\kappa_n} , \qquad (1.4)$$

where κ_n are called *cumulants* (compare with (1.3). From (1.4) one has

$$\boxed{\kappa_n = \frac{1}{i^n}\frac{d^n}{dk^n}(\ln \hat{p}(k))\Big|_{k=0}} . \qquad (1.5)$$

Consequences.

$$\kappa_1 = \mu_1 ,$$

$$\kappa_2 = \mu_2 - \mu_1^2 = \sigma^2$$

$$\kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$$

$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$$

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4. Canonical probability densities

4.1. Gaussian PDF (normal PDF) as arising from the first two non-zero cumulants

Assume κ_1, $\kappa_2 \neq 0$, $\kappa_3 = \kappa_4 = \ldots = 0$. Then, the CF takes the form

$$\ln \hat{p}(k) = ik\kappa_1 - \frac{k^2}{2}\kappa_2 \quad , \quad \hat{p}(k) = \exp\left(ik\mu_1 - \frac{k^2}{2}\sigma^2\right) .$$

Making an inverse Fourier transform, one gets the *Gaussian PDF*

$$
\boxed{g(x;\mu_1,\sigma)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right) \quad \div \quad g(k;\mu_1,\sigma)=\exp\left(ik\mu_1-\frac{k^2}{2}\sigma^2\right)} \tag{1.6}
$$

Indeed,

$$
\hat{p}(k;\mu_1,\sigma)=\int_{-\infty}^{\infty} e^{ikx}\frac{e^{-\frac{(x-\mu_1)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}dx = \text{<introduce } y=\frac{x-\mu}{\sigma}\text{>} =
$$

$$
=\frac{e^{ik\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(y^2-2ik\sigma y-(k\sigma)^2+(k\sigma)^2\right)\right]dy=
$$

$$
=\frac{e^{i\mu k-\sigma^2k^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-ik\sigma)^2} = e^{i\mu k-\sigma^2k^2/2} .
$$

Important in data analysis ! The higher cumulants κ_n (n = 3,4,) give the quantitative measures how an arbitrary distribution deviates from the *symmetric* Gaussian distribution, $\kappa_1 = 0$.

- *skewness*

$$
K_3=\frac{\mu_3}{\sigma^3} \quad ;
$$

- *kurtosis*

$$
K_4=\frac{\kappa_4}{\sigma^4}=\frac{\mu_4-3\mu_2^2}{\sigma^4}=\frac{\mu_4}{\sigma^4}-3 \quad .
$$

Moments:

$$
\langle x\rangle=\mu \qquad \left\langle x^2\right\rangle=-\left.\frac{d^2\hat{p}(k)}{dk^2}\right|_{k=0}=\sigma^2+\mu^2
$$

$$
\left\langle\left(x-\langle x\rangle\right)^2\right\rangle=\left\langle\left(x^2-2x\langle x\rangle+\langle x\rangle^2\right)\right\rangle=\left\langle x^2\right\rangle-\langle x\rangle^2=\sigma^2
$$

4.2. Laplace PDF (double exponential PDF)

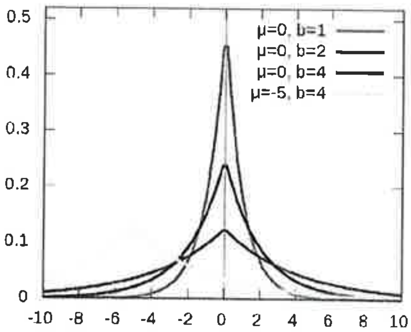
$$
\boxed{p(x;\mu,b)=\frac{1}{2b}e^{-\frac{|x-\mu|}{b}} \quad \div \quad \hat{p}(k;\mu,b)=\frac{e^{i\mu k}}{1+b^2k^2}} \tag{1.7}
$$

Remarks. 1) Kurtosis (Sometimes defined as) $K=\frac{\mu_4}{\sigma^4}$ (no "-3")

greater likelihood of extreme events →

- Leptokurtic distribution $K > 3$
- Mesokurtic $K = 3$
- Platykurtic $K < 3$

2) Risk-seeking investors: focus on investments whose returns follow leptokurtic distr.



5.

$$\hat{p}(k;\mu,b)=\frac{1}{2b}\int_{-\infty}^{\infty} e^{ikx-\frac{|x-\mu|}{b}}dx =< y=\frac{x-\mu}{b} >=$$

$$=\frac{e^{ik\mu}}{2}\left\{\int_0^{\infty} e^{(ikb-1)y}dy+\int_0^{\infty} e^{(-ikb-1)y}dy\right\}=\frac{e^{ik\mu}}{2}\left\{\frac{1}{1-ikb}+\frac{1}{1+ikb}\right\}=\frac{e^{i\mu k}}{1+b^2k^2} .$$

- *First moment*

$$\langle X\rangle=\frac{1}{i}\frac{d\hat{p}(k)}{dk}\bigg|_{k=0}=\ldots=\mu ,$$

- *Second moment*

$$\langle X^2\rangle=-\frac{d^2\hat{p}(k)}{dk^2}\bigg|_{k=0}=\mu^2+2b^2 ,$$

- *Variance*

$$\sigma^2=\left\langle(X-\langle X\rangle)^2\right\rangle=\langle X^2\rangle-\langle X\rangle^2=2b^2 .$$

Exercise. Skewness and kurtosis.

Q. Leptokurtic or Platykurtic ?...

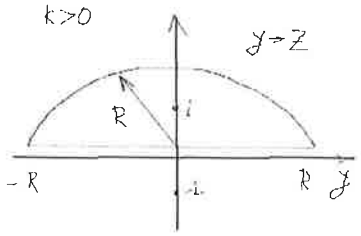
4.3. Cauchy PDF (Cauchy-Lorentz PDF)

$$\boxed{p(x;\mu,b)=\frac{1}{\pi}\frac{b}{(x-\mu)^2+b^2} \quad \div \quad \hat{p}(k;\mu,b)=e^{i\mu k-b|k|}} \qquad (1.8)$$

a) Characteristic function.

$$\hat{p}(k;\mu,b)=\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{(x-\mu)^2+b^2}dx =< y=\frac{x-\mu}{b} >=\frac{e^{i\mu k}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikby}}{y^2+1}=\frac{e^{i\mu k}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikby}}{(y+i)(y-i)} .$$

Reminder. *Residue Theorem.*
For $k>0$ make a closed contour in the upper half-plane:



$$\int\limits_C \frac{e^{ikbz}}{(z+i)(z-i)}dz = 2\pi i\,\mathrm{Res}\{z=i\} = 2\pi i\frac{e^{-bk}}{2i} = \pi e^{-bk}, \quad k>0\,,$$

$$\int\limits_C ...dz = \int\limits_{-\infty}^{\infty} ...dy + \int\limits_{arc} ...dz \quad .$$

Due to Jordan's lemma

$$\int\limits_{arc} \frac{e^{ikbz}}{1+z^2}dz \le \pi R \sup_{arc}\left|\frac{e^{ikbz}}{1+z^2}\right| \le \pi R \sup_{arc}\left|\frac{1}{1+z^2}\right| \sim \frac{1}{R} \to 0\,, \text{ since } R\to\infty\,,$$

thus

$$\int\limits_{-\infty}^{\infty} \frac{e^{ibky}}{y^2+1} = \pi e^{-bk} \quad , \quad k>0 \quad ,$$

and

$$\hat{p}(k;\mu,b) = e^{i\mu k - bk} \quad , \quad k>0 \quad .$$

Similarly, for $k<0$ we make a closed contour in the lower half-plane to get

$$\hat{p}(k;\mu,b) = e^{i\mu k + bk} \quad , \quad k<0 \quad ,$$

and finally

$$\hat{p}(k;\mu,b) = e^{i\mu k - b|k|} \quad .$$

$$\langle X \rangle = \frac{b}{\pi}\left\{ \int_0^{\infty} ... + \int_{-\infty}^{0} ... \right\} = \infty - \infty$$

undefined
not defined
(does not exist)

b) moments
Put $\mu = 0$ for simplicity

$$\langle X^2 \rangle = \int_{-\infty}^{0} ... + \int_{0}^{\infty} ... = \infty + \infty = \infty$$

infinite

The mean

$$\langle X \rangle = \int\limits_{-\infty}^{\infty} xp(x)dx = \frac{b}{\pi}\int\limits_{-\infty}^{\infty}\frac{x}{x^2+b^2}dx = \frac{b}{\pi}\lim_{\substack{L_1\to-\infty\\L_2\to\infty}}\int\limits_{L_1}^{L_2}\frac{x}{x^2+b^2}dx =$$

$$= \lim_{\substack{L_1\to-\infty\\L_2\to\infty}} \ln\left(x^2+b^2\right)\Big|_{L_1}^{L_2} = \frac{b}{\pi}\lim_{\substack{L_1\to-\infty\\L_2\to\infty}} \ln\left|\frac{L_2}{L_1}\right| \quad \text{undefined} \quad .$$

~~*All integer moments* $\langle X^n \rangle$, $n=1,2,...,$ *are undefined.*~~

Remark. The Cauchy principal value.

Exercise. Fractional moments for the Cauchy distribution.

4.4. Exponential PDF

Positive r.v. T, PDF $p_T(t)$

a) Laplace transform pair (symbol $\div$ for the Laplace transform pair, same as for the Forier transform pair, tilde ~ to denote Laplace transform)

$$p_T(t) \div \tilde{p}_T(s) \equiv \left\langle e^{-sT} \right\rangle \equiv \mathbf{E}e^{-sT} = \int_0^\infty e^{-st} p_T(t)dt \quad \div \quad p_T(t) = \frac{1}{2\pi i} \int_{Br} e^{st} \tilde{p}_T(s)ds$$

Laplace transform is more convenient than Forier transform for positive r.v.

b) Definition. *Exponential PDF*

$$\boxed{p(t;\lambda) = \begin{cases} \lambda e^{-\lambda t} & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}} \quad . \tag{1.9}$$

c) *Laplace tranform*

$$\tilde{p}(s;\lambda) = \frac{\lambda}{\lambda + s} \quad \div \quad p(t;\lambda)$$

d) *moments*

$$\left\langle T^n \right\rangle = \int_0^\infty t^n p(t;\lambda)dt = (-1)^n \left. \frac{d^n \tilde{p}(s;\lambda)}{dt^n} \right|_{s=0} = \frac{n!}{\lambda^n} \quad .$$

5. Multivariate distributions

5.1. Joint probability density function (PDF, probability distribution)

r.v. $\vec{X} = \{X_1, X_2, ..., X_n\}$ on n-dimensional set Ωn.

a) meaning

$$p_n(x_1, x_2, ..., x_n)\,dx_1 dx_2 ... dx_n =$$
$$= \Pr\{x_1 \le X_1 < x_1 + dx_1; x_2 \le X_2 < x_2 + dx_2; ...; x_n \le X_n < x_n + dx_n\} .$$

b) non-negativity

$$p_n(x_1, x_2, ..., x_n) \ge 0 .$$

c) normalization

$$\int\limits_{\Omega_n} p_n(x_1, x_2, ..., x_n)\,dx_1 dx_2 ... dx_n = 1 .$$

ADD Example. Bivariate Gaussian distribution

5.2. Cumulative distribution function

$$P_n(x_1, x_2, ..., x_n) = \Pr\{-\infty < X_1 \le x_1; -\infty < X_2 \le x_2; ...; -\infty < X_n \le x_n\} .$$

Remark. "-∞" with no loss of generality.

5.3. Marginal probability density (probability distribution)

Subset $X_1, X_2, \ldots, X_s$, $s < n$

$$p_s(x_1, x_2, ..., x_s) = \int p_n(x_1, ..., x_s, x_{s+1}, ..., x_n)\,dx_{s+1} ... dx_n$$

5.4. Conditional probability density

Fix $X_{s+1}, ..., X_n$.
Conditional PDF

$$p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n)$$

Normalization:

$$\int p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n)\,dx_1 dx_2 ... dx_s = 1 .$$

5.5. Bayes' rule

$$p_n(x_1, x_2, ..., x_n) = p_{n-s}(x_{s+1}, ..., x_n)\,p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n) .$$

$$\boxed{p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n) = \frac{p_n(x_1, x_2, ..., x_n)}{p_{n-s}(x_{s+1}, ..., x_n)}} . \qquad (5.1)$$

5.6. Moments

$$\left\langle X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n} \right\rangle = \int x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \, p_n(x_1, x_2, \ldots, x_n) \quad .$$

5.7. Characteristic function

$$\hat{p}_n(k_1, k_2, \ldots, k_n) \equiv \left\langle \exp(ik_1 X_1 + ik_2 X_2 + \ldots + ik_n X_n) \right\rangle =$$

$$= \int e^{ik_1 x_1 + ik_2 x_2 + \ldots + ik_n x_n} \, p_n(x_1, x_2, \ldots, x_n) =$$

$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(ik_1)^{m_1} (ik_2)^{m_2} \ldots (ik_n)^{m_n}}{m_1! \, m_2! \ldots m_n!} \langle X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n} \rangle$$

5.8. Covariance matrix and correlation coefficients

$$C_{ij} = \left\langle \left(X_i - \langle X_i \rangle\right)\left(X_j - \langle X_j \rangle\right) \right\rangle = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle \quad .$$

Diagonal elements : variances, non-diagonal : covariances.

$$\rho_{ij} = \frac{\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle}{\sqrt{\left(\langle X_i^2 \rangle - \langle X_i \rangle^2\right)\left(\langle X_j^2 \rangle - \langle X_j \rangle^2\right)}} \quad .$$

Exercise. Prove 1) $-1 \le \rho_{ij} \le 1$; 2) if $\rho_{12} = \pm 1 \;\Rightarrow\; X_2 = aX_1 + b, \quad a, b = const$.

5.9. Statistical independence

Definition. Take $n = 2$. X_1 and X_2 are *independent random variables* (i.r.v.) if

– joint PDF factorizes,

$$p_2(x_1, x_2) = p(x_1) p(x_2) \quad ,$$

or equivalently,

– all moments factorize,

$$\left\langle X_1^{m_1} X_2^{m_2} \right\rangle = \left\langle X_1^{m_1} \right\rangle \left\langle X_2^{m_2} \right\rangle \quad ,$$

or equivalently

– characteristic function factorizes,

$$\hat{p}_2(k_1, k_2) = \left\langle \exp(ik_1 X_1 + ik_2 X_2) \right\rangle = \left\langle e^{ik_1 X_1} \right\rangle \left\langle e^{ik_2 X_2} \right\rangle = \hat{p}(k_1) \hat{p}(k_2) \quad .$$

Definition. X_1 and X_2 are *uncorrelated* if $C_{12} = 0$. Weaker than statistical independence.

~~[illegible]~~ irv $X_1, X_2 \Rightarrow$ uncorrelated

uncorr. $X_1, X_2 \not\Rightarrow$ always irv