9.3 Symmetric stable laws: PDFs and moments (continued)

Remark 5. Important!

For any CF/PDF small k-behavior of $\hat{p}(k) \leftrightarrow$

$\leftrightarrow$ large-x behavior of $p(x)$

$$\hat{l}_\alpha(k) = e^{-\sigma^\alpha |k|^\alpha}, \quad 0<\alpha<2 \; ; \; \alpha = 2 \text{ Gaussian CF}$$

$$k \to 0 \quad \hat{l}_\alpha(k) \sim 1 - \sigma^\alpha |k|^\alpha + \dots$$

$$|x| \to \infty \quad l_\alpha(x) \sim C_1(\alpha) \frac{\sigma^\alpha}{|x|^{1+\alpha}}, \quad C_1(\alpha) = \frac{1}{\pi} \sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)$$

$$\hat{l}_\alpha(k) = \int_{-\infty}^{\infty} e^{ikx} l_\alpha(x)\,dx = 2\int_0^\infty \cos(kx)\, l_\alpha(x)\,dx = 1 - 2\int_0^\infty \left(1-\cos(kx)\right) l_\alpha(x)\,dx \sim$$

$$\int_{-\infty}^{\infty} l_\alpha(x)\,dx = 2\int_0^\infty l_\alpha(x)\,dx = 1$$

$$\sim 1 - 2C_1(\alpha)\sigma^\alpha \int_0^\infty \frac{1-\cos kx}{x^{1+\alpha}}\,dx \Rightarrow$$

$$\int_0^\infty \frac{1-\cos(kx)}{x^{1+\alpha}}\,dx = \{y = |k|x\} = |k|^\alpha \int_0^\infty \frac{1-\cos y}{y^{1+\alpha}}\,dy = \frac{|k|^\alpha}{\alpha}\int_0^\infty \frac{\sin y}{y^\alpha}\,dy =$$

$$= \frac{|k|^\alpha}{2C_1(\alpha)}$$

$$\Rightarrow \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\hat{l}_\alpha(k) \sim 1 - \sigma^\alpha |k|^\alpha}$$

Opposite direction: $1 - \sigma^\alpha |k|^\alpha \longrightarrow C_1(\alpha)\dfrac{\sigma^\alpha}{|x|^{1+\alpha}}$

Requires the theory of improper integrals

9.4 Generalized Central Limit Theorem for symmetric probability laws

$X_1, X_2, \dots X_n$ iid $p_X(x)$, $\quad p_X(x) = p_X(-x)$, $\quad \langle X^2 \rangle = \infty$

Let $p(x) \underset{x\to\pm\infty}{\sim} \dfrac{A}{|x|^{1+\alpha}}$, $\quad 0<\alpha<2$, $A = \text{const}$ $\Rightarrow \langle X^2 \rangle = 2\int_0^\infty x^2 \underbrace{p(x)}_{1/x^{\alpha-1}}\,dx = \infty$

divergence at $x = \infty$

Properly normalized sum (Reminder: CLT $Y = \frac{1}{\sqrt{n}} \sum_{j=1}^n X_j$)

$$Y_n = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} X_j$$

Theorem $\quad p_{Y_n}(y) \underset{n\to\infty}{\longrightarrow} l_\alpha(y) \sim \dfrac{A}{|y|^{1+\alpha}}, \quad y \to \infty, \quad 0<\alpha<2$

Two important statements

1. Only asymptotic behavior of $p_X(x)$ at $x \to \infty$ is important for convergence to stable law.
2. The limiting stable PDF $l_\alpha(y)$ has exactly the same

for convergence to stable

2. The limiting stable PDF $\ell_\alpha(y)$ has exactly the same asymptotic behavior as $p_X(x)$

<u>Remark</u> $\ell_\alpha(y) \doteq \hat{\ell}_\alpha(k) = e^{-\sigma_Y^\alpha |k|^\alpha}$

$$A = \frac{\sigma_Y^\alpha}{\pi} \sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)$$

$$\boxed{\sigma_Y = \left[\frac{\pi A}{\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha)}\right]^{1/\alpha}}$$

Summary of GCLT

$X_1, \dots, X_n$ iid $p_X(x)$

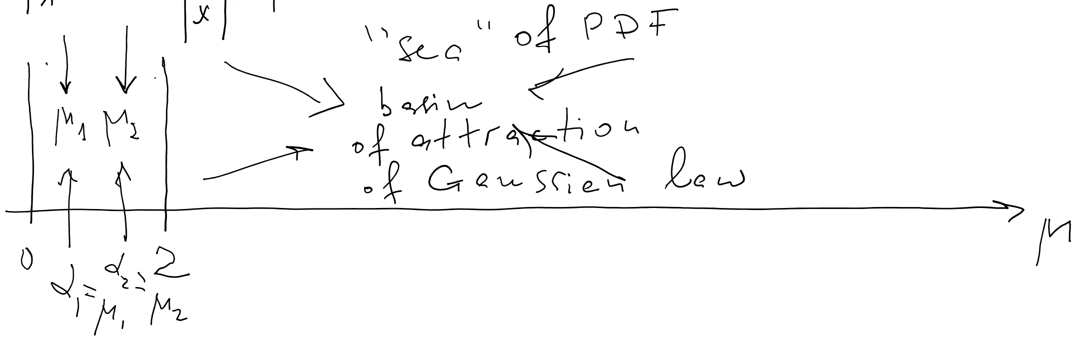
CLT $\to$ basin of attraction of the Gaussian law $p_X(x)$ with finite variance ($\alpha = 2$)

GCLT $\to$ basin of attraction of α-stable laws $p_X(x)$ with with <u>infinite</u> variance and <u>the same asymptotic behavior as α-stable law</u>, $0 < \alpha < 2$

<u>Remark</u> $p(x) \sim \frac{1}{x^3}$? $\quad Y_n = \frac{1}{\sqrt{n \ln n}} \sum_{j=1}^{n} X_j$

$$p_{Y_n}(y) \xrightarrow[n\to\infty]{} \text{Gaussian law}$$

$p_X(x) \sim \frac{A}{|x|^{1+\mu}}$, $0 < \mu < 2$, $x \to \infty$ $\Rightarrow$ $p_Y(y) \xrightarrow[n\to\infty]{}$ stable law, $\alpha = \mu$



(9.5) Linear transformation

r.v. $\ell_\alpha(x) \doteq \hat{\ell}_\alpha(k) = e^{-|k|^\alpha}$ stable analogue of the standard normal law ($\sigma = 1$)

$$\underbrace{Y = \sigma X + \mu}$$

$$\hat{p}_Y(k) = \langle e^{ikY} \rangle = e^{i\mu k}\, \hat{\ell}_\alpha(\sigma k) \doteq \ell_\alpha(y) = \ell_\alpha(y = \sigma x + \mu) \equiv$$

$$\hat{p}_Y(k) = \langle e^{ikY} \rangle = e^{i\mu k}\, \hat{l}_\alpha(\sigma k) \doteq l_\alpha(y) = l_\alpha(y = \sigma x + \mu) \equiv$$
$$\equiv l_\alpha(y; \sigma, \mu)$$

Terminology σ scale parameter $\longleftrightarrow$ standard deviation at $\alpha = 2$

μ shift parameter

$\langle Y \rangle = \mu \leftrightarrow \mu$ = mean value if $1 < \alpha \leq 2$

$\mu = 0 \implies p_Y(y) = p_Y(-y)$ symmetric PDF

<u>Important remark</u> μ and σ can be eliminated by a simple linear transformation

$$\boxed{l_\alpha(y; \sigma, \mu) = \frac{1}{\sigma}\, l_\alpha\left(\frac{y-\mu}{\sigma}; 1, 0\right)} \qquad (A)$$

Take (A), $\times e^{iky}$ and $\int_{-\infty}^{\infty} dy$:

$$\text{lhs of (A)} = \int_{-\infty}^{\infty} e^{iky}\, l_\alpha(y; \sigma, \mu)\, dy = \underline{e^{i\mu k - \sigma^\alpha |k|^\alpha}}$$

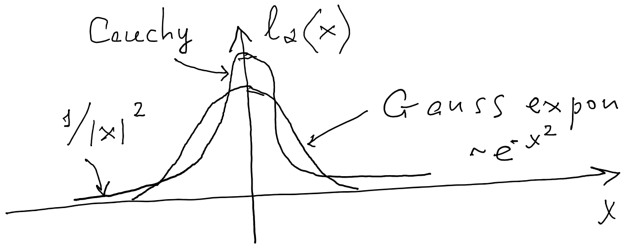
$$\text{rhs of (A)} = \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{iky}\, l_\alpha\left(\frac{y-\mu}{\sigma}; 1, 0\right) dy = \left\{ x = \frac{y-\mu}{\sigma} \right\} =$$
$$= \int_{-\infty}^{\infty} e^{ik(\sigma x + \mu)}\, l_\alpha(x; 1, 0)\, dx = e^{i\mu k}\int_{-\infty}^{\infty} e^{ik\sigma x}\, l_\alpha(x; 1, 0)\, dx =$$
$$= \underline{e^{i\mu k}\, \hat{l}_\alpha(\sigma k)}$$

9.6 PDF plots

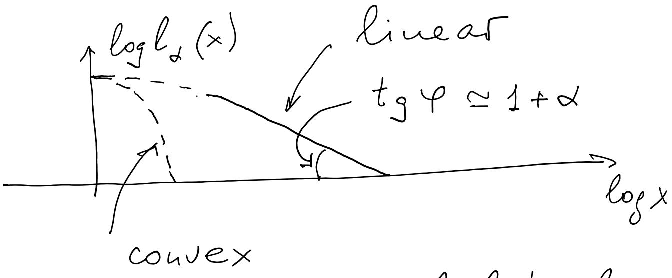
$$l_\alpha(x=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, \hat{l}_\alpha(k)\, dk \,\Big|_{x=0} = \frac{2}{2\pi}\int_0^\infty e^{-k^\alpha}\, dk =$$
$$= \left\{ y = k^\alpha,\ k = y^{1/\alpha},\ dk = \frac{1}{\alpha} y^{\frac{1}{\alpha}-1} dy \right\} = \frac{1}{\pi\alpha}\int_0^\infty e^{-y}\, y^{\frac{1}{\alpha}-1}\, dy =$$
$$= \frac{\Gamma(1/\alpha)}{\pi\alpha}$$

$$\alpha = 2 \qquad l_2(0) = \frac{\Gamma(1/2)}{2\pi} \quad \frac{1}{2\sqrt{\pi}}$$

$$\alpha = 1 \qquad l_1(0) = \frac{1}{\pi}$$



$$l_\alpha(x) \sim \frac{1}{x^{1+\alpha}}, \quad x \to \infty$$



(9.7) General form of CF for α-stable probability laws

$$\boxed{\hat{l}_{\alpha,\beta}(k;\sigma,\mu) = \exp\left[i\mu k - \sigma^\alpha |k|^\alpha \left(1 - i\beta\, \mathrm{sgn}(k)\, \omega(k,\alpha)\right)\right]}$$

$0 < \alpha \le 2$ index of stability (Lévy index)

$-1 \le \beta \le 1$ parameter of asymmetry

μ shift parameter, real

σ scale parameter, positive

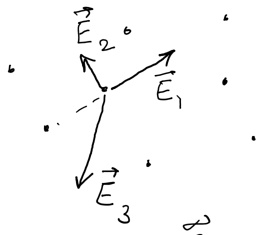
- μ and σ are not so important as α and β: can be removed by linear transformation, see Eq. (A)
- $\beta = 0$ for symmetric laws $l_\alpha(x) = l_\alpha(-x)$

$$\omega(k,\alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \alpha \ne 1, \ 0 < \alpha < 2 \\ -\frac{2}{\pi} \ln|k|, & \alpha = 1 \end{cases}$$

$\alpha = 2 \Rightarrow \beta$ irrelevent

(9.8) Holtzmark distribution (1919)

System of identical charged particles distributed randomly in space



$$\vec{E}_i = \frac{e\vec{r}_i}{r_i^3} \qquad n = \frac{N}{V} = \text{const}$$

$$\vec{E} = \vec{E}_1 + \dots \vec{E}_n, \quad p(\vec{E}) = ?$$

$$p(\vec{E}) = \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{E}}\, \hat{p}_E(\vec{k}) \frac{d\vec{k}}{2\pi}$$

Rigorous calculation $\hat{p}_E(\vec{k}) = e^{-a|\vec{k}|^{3/2}}$

$\hat{p}(\vec{E}) \sim \dfrac{1}{|\vec{E}|^{5/2}}$ 3-dimensional symmetric α-stable law with $\alpha = 3/2$

Similar examples: systems with long-range interactions

- gravity field of stars $\alpha = 3/2$
- electric field of dipoles $\alpha = 1$
- velocity field of point vortices in fully developed turbulence $\alpha = 3/2$

. . . .

Discussion Assignment 1

\# (1.2) Skewness & Kurtosis $K_3 = \dfrac{\bar{\mu}_3}{\sigma^3}$, $K_4 = \dfrac{\bar{\mu}_4}{\sigma^4}$

$$\bar{\mu}_n = \langle (x - \mu_1)^n \rangle \qquad \mu_1 = \langle x \rangle$$

At the lecture $\mu = \langle x \rangle = 0$

$$\langle x \rangle_G = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx =$$

$$= \int_{-\infty}^{\infty} (x - \mu + \mu) \dots dx = \{x - \mu = y\} =$$

$$\int \dots 1 \dots \exp\left[-\frac{y^2}{\dots}\right] dy + \mu \int_{\dots}^{\infty} \frac{e^{-y^2/2\sigma^2}}{\dots} dy = \mu$$

$$= \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{y^2}{2\sigma^2}\right] dy + \mu \int_{-\infty}^{\infty} \frac{e^{-y/2\sigma}}{\sqrt{2\pi\sigma^2}} dy = \mu$$

$$Var\{x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = \sigma^2$$

Check. $K_4 = \frac{\bar{\mu}_4}{\sigma^4} = \frac{1}{\sigma^4} \int_{-\infty}^{\infty} (x-\mu)^4 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx =$

$= 3$

Excessive kurtosis $K_{ex4} = K_4 - 3$

<u>Laplece</u> $p_L(x) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$

Norm $\int_{-\infty}^{\infty} p_L(x)\, dx = \{y = x-\mu\} = \frac{1}{2b} \int_{-\infty}^{\infty} e^{-|x|/b} dx = \frac{2}{2b} \int_0^{\infty} e^{-x/b} dx =$

$= \{y = \frac{x}{b}\} = \int_0^{\infty} e^{-y} dy = 1$

1st moment $\int_{-\infty}^{\infty} x\, p_L(x)\, dx = \int_{-\infty}^{\infty} (\underbrace{x-\mu}+\mu) \frac{1}{2b} e^{-|x|/b} =$

$= \underbrace{\mu \frac{1}{2b} \int_{-\infty}^{\infty} e^{-|x|/b} dx}_{=1} = \mu$

$$Var\{x\} = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} dx = \frac{1}{2b} \int_{-\infty}^{\infty} x^2 e^{-|x|/b} dx =$$

$$= \frac{1}{b} \int_0^{\infty} x^2 e^{-x/b} dx = \frac{1}{b} \left[\frac{\partial^2}{\partial\alpha^2} \int_0^{\infty} e^{-\alpha x}\right]_{\alpha=1/b} = \frac{1}{b} \left[\frac{\partial^2}{\partial\alpha^2} \frac{1}{\alpha}\right]_{\alpha=1/b}$$

$$= \frac{1}{b} \left[\frac{\partial}{\partial\alpha}\left(-\frac{1}{\alpha^2}\right)\right]_{\alpha=1/b} = \frac{2}{b} \frac{1}{\alpha^3}\Big|_{\alpha=1/b} = 2b^2 = \sigma^2$$

$$\bar{\mu}_4 = \frac{1}{2b} \int_{-\infty}^{\infty} x^4 e^{-|x|/b} dx = \frac{2}{2b} \int_0^{\infty} x^4 e^{-x/b} dx = \frac{1}{b} \left[\frac{\partial^4}{\partial\alpha^4} \int_0^{\infty} e^{-\alpha x} dx\right]_{\alpha=1/b} =$$

$$= \frac{1}{b} \left[\frac{\partial^4}{\partial\alpha^4} \frac{1}{\alpha}\right]_{\alpha=1/b} = 24b^4$$

$$K_4 = \frac{24b^4}{4b^4} = 6$$

Problem 4 $\quad \rho = \dfrac{cov(X,Y)}{\sqrt{Var[X]\,Var[Y]}}$

$\langle X\rangle = \langle Y\rangle = 0 \quad , \quad Var[X] = \langle X^2\rangle \; , \quad Var[Y] = \langle Y^2\rangle$

a) start from $\quad \left\langle \left(\dfrac{X}{\sigma_X} \pm \dfrac{Y}{\sigma_Y}\right)^2 \right\rangle \geqslant 0$

$$\underbrace{\frac{\langle X^2\rangle}{\sigma_X^2}}_{=1} \pm \underbrace{\frac{2\langle XY\rangle}{\sigma_X\sigma_Y}}_{=2\rho} + \underbrace{\frac{\langle Y^2\rangle}{\sigma_Y^2}}_{=1} \geqslant 0$$

$$2 \pm 2\rho \geqslant 0 \implies -1 \leq \rho \leq 1$$

b) $\underbrace{Y = aX}$

$$\rho = \frac{\langle XY\rangle}{\sqrt{\langle X^2\rangle\langle Y^2\rangle}} = \frac{a\langle X^2\rangle}{|a|\langle X^2\rangle} = \operatorname{sign} a = \pm 1$$

Problem 6 $\quad p_Z(z) = \displaystyle\iint_{-\infty}^{\infty} p(x,y)\,\delta(z - f(x,y))\,dx\,dy$

$$Z = \frac{X}{Y} \qquad p_Z(z) = \iint_{-\infty}^{\infty} \underbrace{p(x,y)}_{p(x)p(y)}\,\delta\!\left(z - \frac{x}{y}\right) dx\,dy = \left\{ \delta(ax) = \frac{1}{|a|}\delta(x) \right\}$$

$$= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\; p_X(x)\,p_Y(y)\,\delta\!\left[\frac{1}{|y|}(x - zy)\right] = \int_{-\infty}^{\infty} dy\,|y|\,p_Y(y) \int_{-\infty}^{\infty} dx\; p_X(x)\,\delta(x - zy) \Rightarrow$$

$$\Rightarrow \boxed{p_Z(z) = \int_{-\infty}^{\infty} p_Y(y)\,p_X(zy)\,|y|\,dy}$$

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right), \quad p_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}}\, e^{-x^2/(2\sigma_Y^2)}$$

$$p_Z(z) = \frac{1}{2\pi\sigma_X\sigma_Y} \int_{-\infty}^{\infty} e^{-y^2/2\sigma_Y^2 - z^2y^2/2\sigma_X^2}\,|y|\,dy = \left\{ A = \frac{z^2}{2\sigma_X^2} + \frac{1}{2\sigma_Y^2} \right\} =$$

$$= \frac{1}{\pi\sigma_X\sigma_Y} \int_0^{\infty} y\, e^{-Ay^2}\,dy = \left\{ \eta = y^2,\; d\eta = 2y\,dy \right\} =$$

$\int^{\infty} e^{-A\eta}$ … $\quad \dfrac{1}{\quad\quad}$ ~~$= \{\sigma_X^2 = \sigma_Y^2 = 1\} =$~~

$\cancel{\pi \sigma_x \sigma_y \cdot 1}$

$$= \frac{1}{2\pi \sigma_x \sigma_y} \int_0^\infty e^{-A\eta} \, d\eta = \frac{1}{2\pi \sigma_x \sigma_y A(z)} = \{ \sigma_x^2 = \sigma_y^2 = 1 \} =$$

$$= \frac{1}{2\pi \left(\frac{z^2}{2} + \frac{1}{2} \right)} = \frac{1}{\pi (1 + z^2)} = \text{Cauchy}$$