Lecture 4. Chapter 4. Bloch Sphere

- Useful tool to represent actions of various q.-mech. operators on a spinor;
- Link between abstract concept of spinor and more intuitive way of thinking of spin as being associated with self-rotation

  picture

  concept of spinor $\Leftrightarrow$ self-rotation
- Useful for describing action of magnetic field on the spin (next lecture)

  Useful for spin in magnetic field
- Invoked in discussions of spin-based quantum computing

4.1 Spinor and qubit (= quantum bit)

- "Classical" computers: process classical binary bits 0 and 1;
- Quantum comp: process quantum bits = coherent superpositions of both 0 and 1

It was realized long time ago:

- Spin polarization of electron is ideal physical implementation of a qubit ← polarization can survive in a spin coherent superposition of two orthogonal states for (relatively) long time

These states represent classical bits 0 and 1

- From previous lecture:
  2-component wavefunction representing an arbitrary spin state

$$[\psi] = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \phi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \phi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \phi_1 |+\rangle_z + \phi_2 |-\rangle_z \quad , \qquad (*)$$

$$|\phi_1|^2 + |\phi_2|^2 = 1$$

⇒ Any spin state can be expressed as coherent superposition of +z and -z-polarized states (mutually orthogonal) that can represent classical bits 0 and 1

- ϕ_1, ϕ_2 complex quantities (magnitude and phase)
- Coherent superposition state : maintain phase relationship between ϕ_1 and ϕ_2 until the qubit is "read" and collapses to either +z or -z-polarized states (with probabilities $|\phi_1|^2$ and $|\phi_2|^2$)
- +z-polarized state → bit 0 → spinor $|0\rangle$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ Dirac's ket notation

-z-polarized state → bit 1

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eq $(*)$ $\Rightarrow$

$$|\chi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\alpha}_{\substack{|\alpha|^2 \text{ prob. to} \\ \text{find spin} \\ \text{in state } |0\rangle}}|0\rangle + \underbrace{\beta}_{\text{in state } |1\rangle \ |\beta|^2}|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Spinor $|\chi\rangle$ is a qubit : coherent superposition of bits 0 and 1

- α, β : continuous complex ~~quantum~~ variables

Reminder: geometrical representation of complex ~~variables~~ numbers in the complex plane

Question: ~~E~~ Useful representation of Spinors or qubits ?

Answer: Bloch sphere

(4.2) Preliminaries. Eigenspinors of $(\vec{S}\,\vec{n})$

- Reminder $\quad \vec{S} = \frac{\hbar}{2}\vec{\sigma} \qquad (1)$

  S_x, S_y, S_z satisfy comm. rules (similar to orbital angular momentum $\vec{L}$)

- measure spin component along direction $\vec{n}$ $\Rightarrow$ eigenvalues of $\vec{S}\cdot\vec{n}$ (e.v.)

$$\Downarrow$$

(a) Fact : e.val $= \pm \frac{\hbar}{2}$ irrespective of direction

Proof.

Exercise $\quad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b}) + \vec{a}\cdot\vec{b}\,I \quad (2)$

$\vec{a}, \vec{b}$ vectors in 3D.

Take $\vec{a} = \vec{b} = \vec{n} \Rightarrow (\vec{\sigma}\,\vec{n})^2 = I$

$$\Downarrow$$

eigenvalues of $(\vec{\sigma}\cdot\vec{n})$ are ± 1

$$\Downarrow \quad (1)$$

e.val of $(\vec{S}\,\vec{n})$ are $\pm\frac{\hbar}{2}$

Measurement of spin ang. momentum along any arbitrary axis yields $\pm\hbar/2$.

(b) to find e.vec

Consider

$$\text{Step 1.}\quad (\vec{\sigma}\vec{n})\left[\frac{1}{2}(I \pm \vec{\sigma}\vec{n})|\psi\rangle\right] =$$

qubit

$$= \frac{1}{2}(\vec{\sigma}\vec{n})|\psi\rangle \pm \frac{1}{2}\underbrace{(\vec{\sigma}\vec{n})^2}_{I}|\psi\rangle =$$

$$= \pm\left[\frac{1}{2}(I \pm \vec{\sigma}\vec{n})|\psi\rangle\right] \qquad (3)$$

$$\Downarrow$$

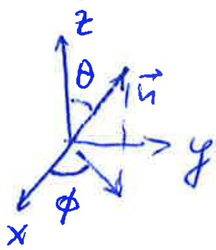
for $\forall\, |\psi\rangle$, $\frac{1}{2}(I \pm \vec{\sigma}\vec{n})|\psi\rangle$ are eigvec of $\vec{\sigma}\vec{n}$ with eigval ± 1.

Step 2. Make use of identity

$$\frac{1}{2}(I \pm \vec{\sigma}\vec{n}) = \frac{1}{2}\left[I \pm \sigma_z n_z \pm \frac{1}{2}(\sigma_x + i\sigma_y)(n_x - in_y) \pm \right.$$

$$\left. \pm \frac{1}{2}(\sigma_x - i\sigma_y)(n_x + in_y)\right] \qquad (4)$$

In spherical coordinates



$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$n_x \pm in_y = \sin\theta\, e^{\pm i\phi}$$ plug in (4)

$$\frac{1}{2}(I \pm \vec{\sigma}\vec{n}) = \frac{1}{2}\left[I \pm \cos\theta\,\sigma_z \pm \frac{1}{2}\left\{\sin\theta\, e^{-i\phi}\sigma_+ \pm \sin\theta\, e^{i\phi}\sigma_-\right\}\right] \qquad (5)$$

$$\sigma_+ = \sigma_x + i\sigma_y, \quad \sigma_- = \sigma_x - i\sigma_y$$

Remind. 1) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

$$|\alpha|^2 + |\beta|^2 = 1 \quad , \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2) Lecture 3

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Step 3. Act with operators (5) on ket $|0\rangle$

$$\frac{1}{2}\left(I+\vec{\sigma}\vec{n}\right)|0\rangle =$$

$$= \frac{1}{2}\left[I+\cos\theta\cdot\sigma_z + \frac{1}{2}\sin\theta\left\{e^{-i\phi}\sigma_+ + e^{i\phi}\sigma_-\right\}\right]\begin{pmatrix}1\\0\end{pmatrix} \quad (6)$$

- $I\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

- $\sigma_z\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

- $\sigma_+\begin{pmatrix}1\\0\end{pmatrix}=(\sigma_x+i\sigma_y)\begin{pmatrix}1\\0\end{pmatrix}=$

$$=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=$$

$$=\begin{pmatrix}0\\1\end{pmatrix}+i\begin{pmatrix}0\\i\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

- $\sigma_-\begin{pmatrix}1\\0\end{pmatrix}=(\sigma_x-i\sigma_y)\begin{pmatrix}1\\0\end{pmatrix}=$

$$=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=$$

$$=\begin{pmatrix}0\\1\end{pmatrix}-i\begin{pmatrix}0\\i\end{pmatrix}=2\begin{pmatrix}0\\1\end{pmatrix}=2|1\rangle$$

Combine in (6)

(4.8)

- $\underline{\frac{1}{2}(I + \vec{\sigma}\vec{n})|0\rangle} =$

$$= \frac{1}{2}|0\rangle + \frac{1}{2}\cos\theta|0\rangle + \frac{1}{4}\sin\theta e^{i\phi}\cdot 2|1\rangle =$$

$$= \left\{ \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \right\}$$

$$= \frac{1}{2}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)|0\rangle + \frac{1}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)|0\rangle$$

$$+ \sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}e^{i\phi}|1\rangle =$$

$$= \underline{\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]} \quad (7)$$

- Similarly,

$$\frac{1}{2}(I - \vec{\sigma}\vec{n}|0\rangle) = \sin\frac{\theta}{2}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{+i\phi}|1\rangle\right] \quad (8)$$

- Normalize (7), (8):

$$\boxed{\begin{aligned} |\xi_n^+\rangle &= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle \\ |\xi_n^-\rangle &= \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{+i\phi}|1\rangle \end{aligned}} \quad (9)$$

~~Andro~~

Remind ~~that~~ Eq (3): $\forall |\chi\rangle$, $\frac{1}{2}(I \pm \vec{\sigma}\vec{n})|\chi\rangle$ e.vecs of $\vec{\sigma}\vec{n}$ with evals ± 1 $\Rightarrow$ $|\xi_n^+\rangle$, $|\xi_n^-\rangle$ are e.vecs (eigenspinors) of $\vec{\sigma}\vec{n}$ with e.val. $+1$, -1 resp.

Remark 1. Easy to check orthogonality

$$\langle \xi_n^+ | \xi_n^- \rangle = 0$$

Remark 2. $|\xi_n^-\rangle$ is obtained from $|\xi_n^+\rangle$

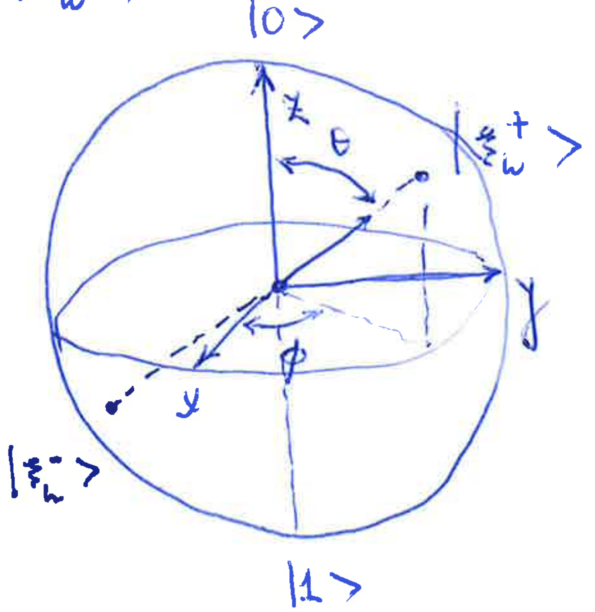
$$|\xi_n^-(\theta,\phi)\rangle = |\xi_n^+(\theta\to\pi-\theta,\ \phi\to\phi+\pi)\rangle$$

$0 \leq \theta < \pi \quad , \quad 0 \leq \phi < 2\pi$

Remark 3. The most general form of the spinor

$$\boxed{|\xi_n^+\rangle = e^{i\delta}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right]} \quad (10)$$

- one-to-one relation between $\vec{n}$ and eigenspin $|\xi_n^+\rangle$



Useful when analyzing action of 2×2 matrices acting on spinor $|\xi_n^+\rangle$

4.3 Expectation values of Pauli matrices

- For electron characterised by $|\xi_n^+\rangle$

$$\langle \xi_n^+ | \sigma_x | \xi_n^+ \rangle = \sin\theta\cos\phi$$

$$\langle \xi_n^+ | \sigma_y | \xi_n^+ \rangle = \sin\theta\sin\phi$$

$$\langle \xi_n^+ | \sigma_z | \xi_n^+ \rangle = \cos\theta$$

Cartesian coordinates of $\vec{n}$

(easy exercise)

Think of unit vector $\vec{n}$ (θ, ϕ) as representation of spinor eigenstate of $\vec{\sigma}\,\vec{n}$ with eigenval $+1$.

- Similarly

$$\langle \xi_n^- | \sigma_x | \xi_n^- \rangle = -\sin\theta\cos\phi$$

$$\langle \xi_n^- | \sigma_y | \xi_n^- \rangle = -\sin\theta\sin\phi$$

$$\langle \xi_n^- | \sigma_z | \xi_n^- \rangle = -\cos\theta$$

Cartesian coordinates of $-\vec{n}$

Any two diametrically opposite points on Bloch sphere : representations of eigenstates of $\vec{\sigma}\,\vec{n}$ with eigenval ± 1.

4.4. Relationship with qubit

Reminder: qubit

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle,$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$

(coherent) superposition of bits 0 and 1

$$|\psi\rangle = \begin{bmatrix} \mathrm{Re}\,\alpha + i\,\mathrm{Im}\,\alpha \\ \mathrm{Re}\,\beta + i\,\mathrm{Im}\,\beta \end{bmatrix} = \begin{bmatrix} |\alpha| e^{i\phi_\alpha} \\ |\beta| e^{i\phi_\beta} \end{bmatrix} =$$

$$= e^{i\phi_\alpha} \begin{bmatrix} |\alpha| \\ |\beta| e^{i(\phi_\beta - \phi_\alpha)} \end{bmatrix}$$

Remind Eq (10): $|\xi_n^+\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\phi}|1\rangle\right]$

$$\gamma = \phi_\alpha$$

$$\phi = \phi_\beta - \phi_\alpha$$

$$\left.\begin{aligned} |\alpha| &= \cos\frac{\theta}{2} \\ |\beta| &= \sin\frac{\theta}{2} \end{aligned}\right\} \quad \tan\frac{\theta}{2} = \frac{|\beta|}{|\alpha|} = \frac{\sqrt{1 - |\alpha|^2}}{|\alpha|}$$

$$\theta = 2\arctan\left(\frac{\sqrt{1 - |\alpha|^2}}{|\alpha|}\right)$$

4.5 Special spinors

(a) North and south poles

Eq (10) again $|\xi_n^+\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\phi}|1\rangle\right]$

$$|0\rangle = |\xi_n^+(\theta = 0, \phi, \gamma)\rangle$$

$$|1\rangle = |\xi_n^+(\theta = \pi, \phi, \gamma)\rangle \quad , \ \phi, \gamma \text{ arbitrary}$$

Orthogonal, diametrically opposite

(b) Reaching the equator

Consider

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ +1 & -1 \end{bmatrix}$$

Hadamard matrix widely used in design of quantum logic gates

$$H = H^T = H^{-1}$$

$$HH^T = I$$

One of two eigenvectors

$$|\gamma\rangle_H = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle =$$

location on Bloch sphere?

$$= \alpha|0\rangle + \beta|1\rangle$$

$$\left.\begin{aligned} \alpha &= e^{i\gamma}\cos\frac{\theta}{2} = \cos\frac{\pi}{8} \\ \beta &= e^{i\gamma}e^{i\phi}\sin\frac{\theta}{2} = \sin\frac{\pi}{8} \end{aligned}\right\} \theta = \frac{\pi}{4}, \ \phi = 0, \ \gamma = 2\pi k$$

4.13

$\Rightarrow$ eigenvec is in x-z plane halfway between the north pole and the point where positive x-axis pierces the Bloch sphere.

Other eigenvec diametrically opposite

4.6 Spin flip matrix

Q. 2×2 matrix M which changes $|\xi_n^-\rangle$ into $|\xi_n^+\rangle$? $|\xi_n^+\rangle = e^{i\gamma}[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle]$

$$|\xi_n^+\rangle = e^{i\gamma}\begin{bmatrix}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix} = M|\xi_n^-\rangle =$$

$$= M e^{i\gamma}\begin{bmatrix}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

Evidently $M = \begin{bmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0\end{bmatrix}$

Check $e^{i\gamma}\begin{bmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0\end{bmatrix}\begin{bmatrix}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi}\end{bmatrix} = \begin{bmatrix}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix}$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi}\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi}\end{bmatrix}$$

$$M = e^{-i\frac{\pi}{2}} P(\phi)\sigma_y P(-\phi)$$

$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi}\end{pmatrix}$ phase shift matrix used in design of quantum logic gates

(4.15)

(4.7) Excursions on the Bloch sphere: Pauli matrices revisited

(a) Useful identity

$$e^{i\theta A} = \cos\theta \cdot I + i\sin\theta \cdot A, \quad A^2 = I, \quad \theta \text{ is Re} \qquad (11)$$

Generalization of Euler formula

$$e^{iz} = \cos z + i\sin z$$

Proof. - Use definition of the function of operator

$$e^{i\theta A} = I + (i\theta)A + \frac{(i\theta)^2 A^2}{2!} + \frac{(i\theta)^3 A^3}{3!} + \dots$$

Remind $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + (-1)^k \frac{\theta^{2k}}{(2k)!}\right) I +$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots + (-1)^k \frac{i\theta^{2k+1}}{(2k+1)!}\right) A$$

Remind $\displaystyle \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

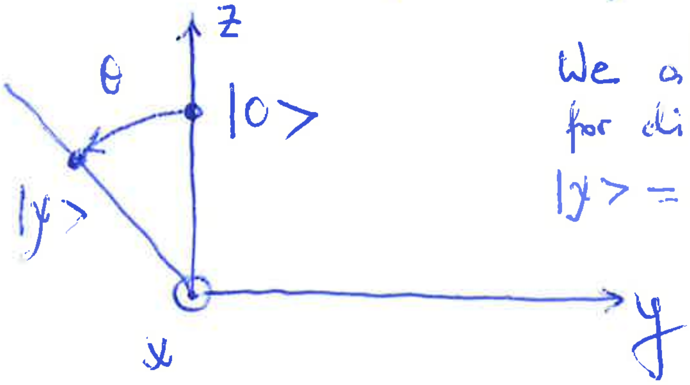
$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$\searrow$ (11)

(b) find unitary matrices allowing to perform rotations on Bloch sphere

Remind. Any spinor

$$|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$



We already know for direction $\vec{n}$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$|\psi\rangle_{new} = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{-i\frac{\pi}{2}}|1\rangle =$$

$$= \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{-i\pi/2}\end{bmatrix} = \begin{bmatrix}\cos\frac{\theta}{2}\\ -i\sin\frac{\theta}{2}\end{bmatrix} \qquad (12)$$

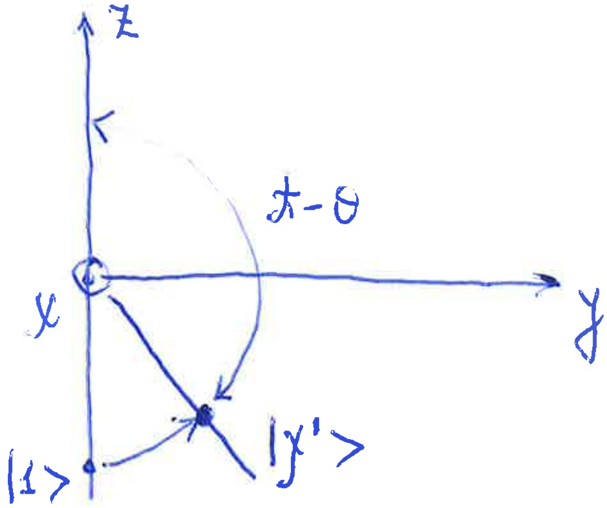
~~In~~ Introduce rotation matrix

$$|\psi\rangle_{new} = R_x(\theta)|0\rangle = \begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}R_{11}\\ R_{21}\end{pmatrix}$$

↑

first column of matrix $R_x(\theta)$.

Apply $R_x(\theta)$ to $|1\rangle$



$$|y'\rangle_{new} = R_x(\theta)|1\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle +$$

$$+ e^{i\pi/2} \sin\left(\frac{\pi-\theta}{2}\right)|1\rangle =$$

$$= \begin{bmatrix} \sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{bmatrix} = i \begin{bmatrix} -i\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix} \qquad (13)$$

$$|y'\rangle_{new} = R_x(\theta)|1\rangle = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix}$$

represents 2nd column of $R_x(\theta)$

- $R_x(\theta)$ must be unitary matrix since norm of spinor unchanged $\quad U^\dagger U = U U^\dagger = I$
- Drop phase ~~phase~~ factor i in (13)

$$R_x(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \qquad (14a)$$

Exercise $R_x^{\dagger}(\theta) R_x(\theta) = I$

Go to (11) $e^{i\theta A} = \cos\theta \cdot I + i\sin\theta \cdot A \Rightarrow$

$$\boxed{R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x}} \qquad (14b)$$

Easy to check with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Similar arguments

$$\boxed{R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} = e^{-i\frac{\theta}{2}\sigma_y}} \qquad (15)$$

$$\boxed{R_z(\theta) = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} = e^{-i\frac{\theta}{2}\sigma_z}} \qquad (16)$$

Conclusion Pauli matrices constitute building blocks to describe rotations on Bloch sphere $\Rightarrow$ called generators of rotations in Hilbert space of spinors associated with spin-1/2 particles

N.D. Mermin, Am. J. Phys 71, 23 (2002) Teaching computer scientist quantum mechanics