Lecture 12. **II. Random Processes**

25 November 2024 17:26

1. Basic definitions

1.1. Deterministic vs statistical description

a) deterministic $\frac{dx}{dt} = -\lambda x$, $x(t=0) = x_0$, $x(t) = x_0 e^{-\lambda t}$

b) $x = f(t)$ is a random function of time

Example. Coordinate of Brownian particle

Random (stochastic) process; t continuous

c) stochastic (random) sequence: t discrete, countable set

d) random function: both b) and c)

1.2. PDF $p(x,t)$

a) meaning $p(x,t)\,dx = \Pr\{x \le x(t) < x + dx\}$

b) properties (i) $p(x,t) \geq 0$

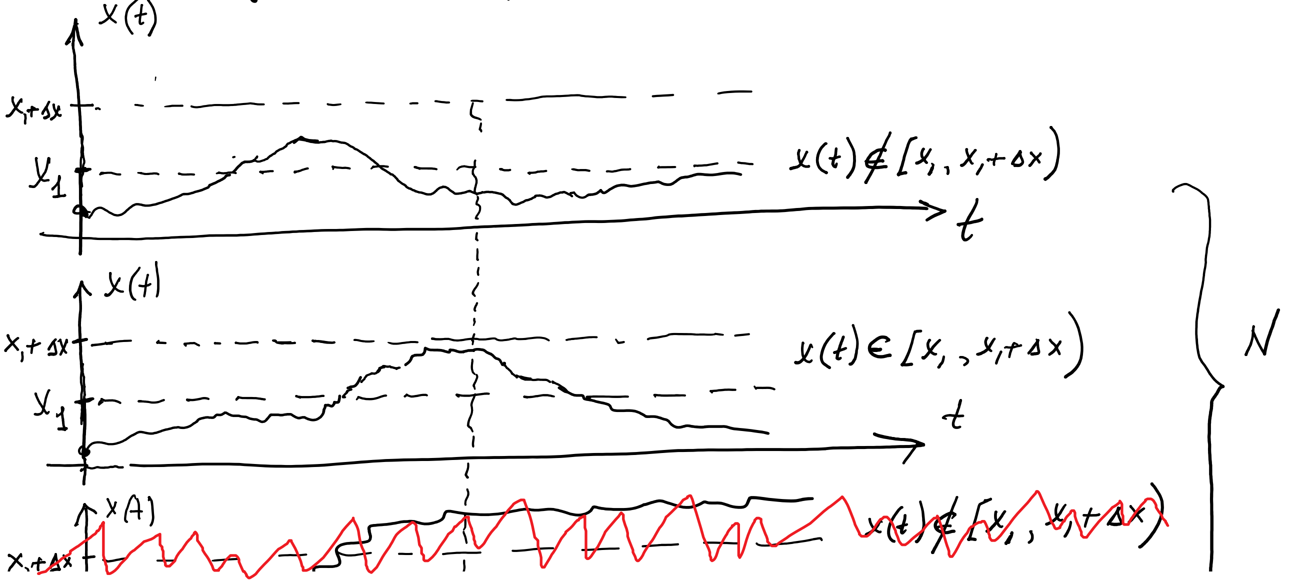
(ii) $\int_{-\infty}^{\infty} p(x,t)\,dx = 1$

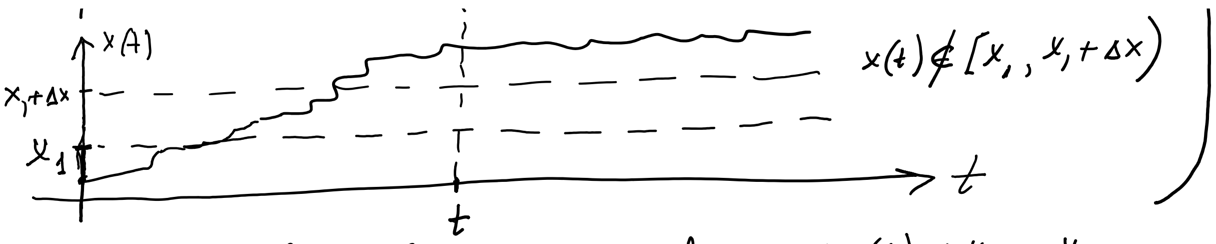
1.3. Statistical ensemble

N Brownian particles, same time, identical conditions, indentical, produce

N trajectories $x^{(j)}(t)$, $j = 1, \dots, N \gg 1$,

(realizations, sample functions)





N_1 : number of realizations with $x_1 \le x(t) < x_1 + \Delta x$

$$P_1 = \Pr\{x_1 \le x(t) < x_1 + \Delta x\} \simeq \frac{N_1}{N}$$

Statistical description is possible if ratio N_1/N is stable.

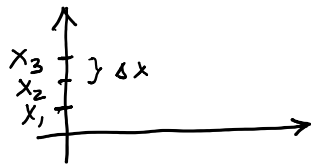
$$\frac{N_1}{N} \xrightarrow[N\to\infty]{} P_1 = p(x = x_1, t)\,\Delta x$$

• PDF $\quad p(x = x_1, t) = \lim_{\substack{N\to\infty \\ \Delta x \to 0}} \frac{1}{\Delta x} \frac{N_1}{N} \qquad (*)$

1.4. Statistical averaging over ensemble of realizations

a) $$\langle x(t)\rangle = \lim_{N\to\infty} \frac{\sum_{j=1}^{N} x^{(j)}(t)}{N}$$

• grouping realizations by intervals



$$x_n \le x(t) < x_n + \Delta x = x_{n+1}$$

• $N_n(t)$ number of realizations in n-th interval $[x_n, x_n + \Delta x]$, $N_n = 0, 1, 2, \ldots$; function of time

$$\underbrace{\langle x(t)\rangle}_{} = \lim_{N\to\infty} \frac{\sum_n x_n N_n(t)}{N} = \lim_{N\to\infty} \sum_n \frac{x_n}{\Delta x} \frac{N_n}{N} \Delta x =$$

$(*)$

$$= \lim_{N\to\infty} \sum_n x_n\, p(x_n, t)\,\Delta x \xrightarrow[\Delta x \to 0]{} \underbrace{\int_{-\infty}^{\infty} x\, p(x,t)\, dx}_{}$$

b) any function $f(x)$ of the random process $x(t)$

$$\langle f(x)\rangle(t) = \int_{-\infty}^{\infty} f(x)\, p(x,t)\, dx$$

$$\equiv$$

$$\langle f(x(t))\rangle$$

c) moments $\mu_n(t) = \langle x^n(t)\rangle = \int_{-\infty}^{\infty} x^n\, p(x,t)\, dx$

d) central moments $\langle (x - \langle x \rangle)^2 \rangle$

variance $\langle (x - \langle x \rangle)^2 \rangle$ (centered second moment)

1.5. Characteristic function

a) definition $\hat{p}(k,t) = \langle e^{ikx(t)} \rangle = \int\limits_{-\infty}^{\infty} e^{ikx} p(x,t)\, dx$

$$\hat{p}(k,t) \div p(x,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-ikx} p(k,t)\, dk$$

b) properties

- $\hat{p}(0,t) = 1$
- $|\hat{p}(k,t)| \leq \int\limits_{-\infty}^{\infty} p(x,t)\, dx = 1$
- $\hat{p}^*(k,t) = \hat{p}(-k,t)$

c) CF as the moment generating function

$$\hat{p}(k,t) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n(t) \Rightarrow \mu_n(t) = \frac{1}{i^n} \left. \frac{d^n \hat{p}(k,t)}{dk^n} \right|_{k=0}$$

d) CF as generating function for the cumulants

$$\ln \hat{p}(k,t) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \varkappa_n(t)$$

1.6. n-point distributions

- Two processes $x(t), y(t)$

$$p_2(x, t_1; y, t_2)\, dx\, dy = Pr\{x \leq x(t_1) < x + dx; y \leq y(t_2) < y + dy\}$$

- Single process at different time instants

a) 2-point PDF

$$p_2(x_1, t_1; x_2, t_2)\, dx_1\, dx_2 =$$
$$= Pr\{x_1 \leq x(t_1) < x_1 + dx_1; x_2 \leq x(t_2) < x_2 + dx_2\}$$

b) 2-point CF

$$\hat{p}_2(k_1, t_1; k_2, t_2) = \iint\limits_{-\infty}^{\infty} e^{i(kx_1 + kx_2)} p_2(x_1, t_1; x_2, t_2)\, dx_1 dx_2$$

c) n-point PDF

$$p_n(x_1, t_1; \ldots; x_n, t_n)\, dx_1 \ldots dx_n =$$

$$= \Pr\{x_j \le x(t_j) < x_j + dx_j\}, \quad j = 1, \ldots, n$$

Q. What does "random process $x(t)$ is fully defined"?

A. We know all p_n for any n and arbitrary chosen $t_1, t_2, \ldots, t_n$

Properties of n-point PDFs

1) symmetry wrt permutations

$$\Pr\{x_j \le x(t_j) < x_j + dx_j, \; j = 1, 2\} =$$

$$= p_2(x_1, t_1; x_2, t_2)\, dx_1 dx_2 = p_2(x_2, t_2; x_1, t_1)\, dx_2 dx_1$$

Event: joint implementation of 2 inequalities

$$x_j \le x(t_j) < x_j + dx_j$$

does not depend on the order how to list them

2) matching condition

k-dim PDF from any n-dim, $k < n$

$$p_k(x_1, t_1; \ldots; x_k, t_k) = \int_{-\infty}^{\infty} p_n(x_1, t_1; \ldots; x_n, t_n)\, dx_{k+1} \ldots dx_n$$

<u>Remark 1.</u> Integration over extra variables $x_{k+1}, \ldots x_n$ leads to the result that does not depend on $t_{k+1}, \ldots, t_n$.

<u>Remark 2</u>. From practical point of view: in most situations $n = 1$ and 2 is enough

Remark 3. Multidimensional random functions.
All the definitions by analogy, e.g.

$$p_n(\vec{r}_1, t_1; \dots; \vec{r}_n, t_n)\, d\vec{r}_1 \dots d\vec{r}_n =$$

$$= \Pr\{\vec{r}_1 \le \vec{r}_1(t_1) < \vec{r}_1 + d\vec{r}_1; \dots; \vec{r}_n \le \vec{r}_n(t) < \vec{r}_n + \underbrace{d\vec{r}_n}\}$$

elementary volume

Remark 4. PDF of deterministic process

$$x = f(t)$$

$$p(x,t) = \delta(x - f(t))$$

1.7. N-point moments

a) $$\langle x(t) \rangle = \int_{-\infty}^{\infty} x\, p(x,t)\, dx$$ mean value

$$\langle (x(t) - \langle x(t) \rangle)^2 \rangle$$ variance

b) $$\langle x(t_1)\, x(t_2) \dots x(t_n) \rangle = \underbrace{\int_{-\infty}^{\infty} \dots \int}_{n} x_1 x_2 \dots x_n\, p_n(x_1, t_1; \dots; x_n, t_n)\, dx_1 \dots dx_n$$

c) 2-point moment, moment of the 2nd order, autocorrelation function (the same process at two different time instants)

$$\langle x(t_1)\, x(t_2) \rangle = \iint_{-\infty}^{\infty} x_1 x_2\, p_2(x_1, t_1; x_2, t_2)\, dx_1\, dx_2 \quad \text{(ACF)}$$

d) autocovariance function (ACVF)
central moment of 2nd order

$$\langle (x(t_1) - \langle x(t_1) \rangle)(x(t_2) - \langle x(t_2) \rangle) \rangle =$$

$$= \langle x(t_1)\, x(t_2) \rangle - \langle x(t_1) \rangle \langle x(t_2) \rangle$$

e) autocorrelation coefficient

$$K(t_1, t_2) = \frac{\langle (x(t_1) - \langle x(t_1)\rangle)(x(t_2) - \langle x(t_2)\rangle)\rangle}{\left[\langle (x(t_1) - \langle x(t_1)\rangle)^2\rangle \langle (x(t_2) - \langle x(t_2)\rangle)^2\rangle\right]^{1/2}}$$

<u>Remark 1</u>. Sometimes in the literature autocorr. coeff. called ACF

<u>Remark 2</u>. $x(t_1)$ and $x(t_2)$ statis~~tical~~lly independent

$$p_2(x_1, t_1; x_2, t_2) = p_1(x_1, t_1)\, p_1(x_2, t_2) \Rightarrow$$

$$\langle x(t_1)\, x(t_2)\rangle = \langle x(t_1)\rangle \langle x(t_2)\rangle \Rightarrow$$

$$\Rightarrow \text{ACVF} = 0 \Rightarrow K(t_1, t_2) = 0$$

<u>Remark 3</u>. Conditional PDFs: later in "Markov processes"