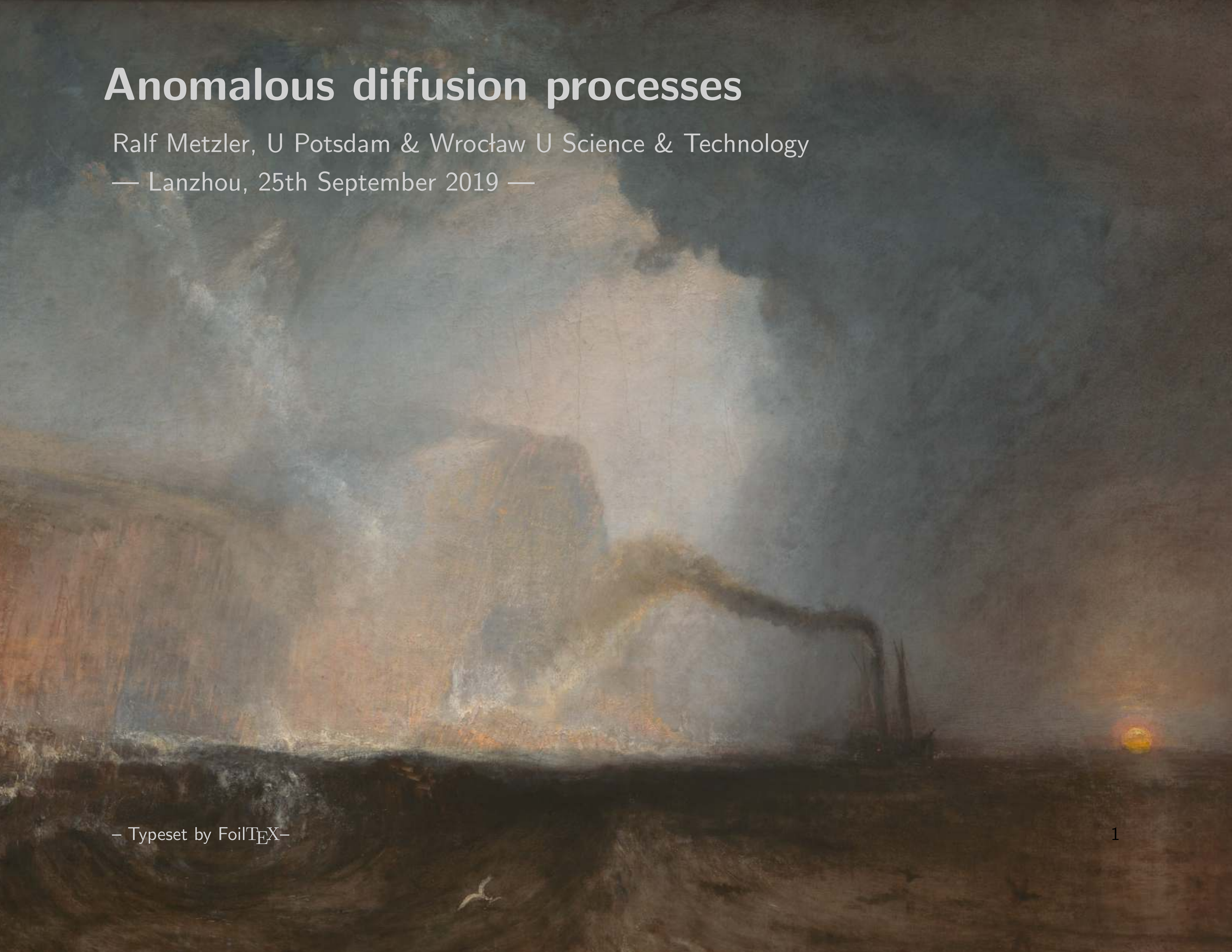
Anomalous diffusion processes

Ralf Metzler, U Potsdam & Wrocław U Science & Technology

— Lanzhou, 25th September 2019 —



Relaxation processes

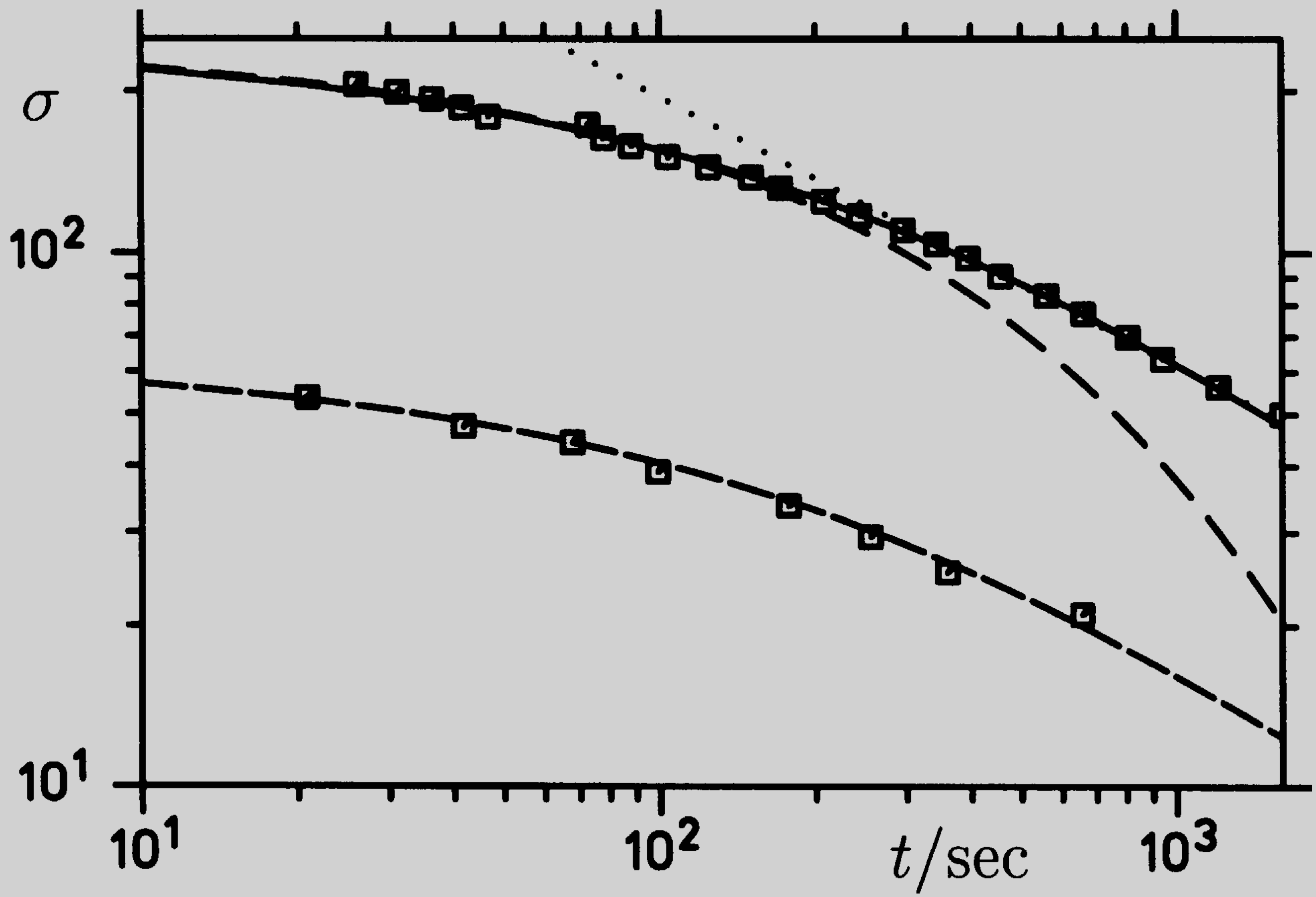
Deborah number:

$$\daleth = \frac{\text{Relaxation time scale}}{\text{Experimental time}}$$

The tar pitch drop experiment (U Queensland):

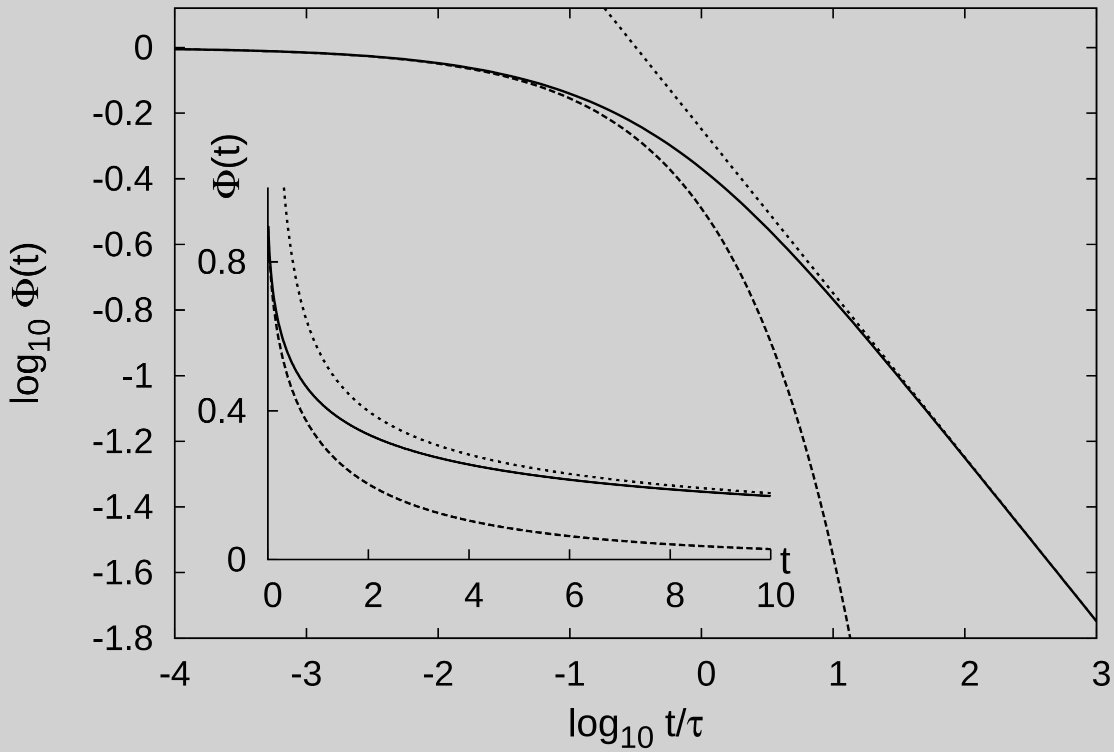
Date	Event
1927	Setup experiment
1930	Stem cut
12/1938	1st drop fell
02/1947	2nd drop fell
04/1954	3rd drop fell
05/1962	4th drop fell
08/1970	5th drop fell
04/1979	6th drop fell
07/1988	7th drop fell
11/2000	8th drop fell

Mittag-Leffler relaxation in rheology



WG Glöckle & TF Nonnenmacher, Macromol (1991)

Mittag-Leffler relaxation



$$E_\alpha\left(-(t/\tau)^\alpha\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(1+\alpha n)} \left(\frac{t}{\tau}\right)^{\alpha n} \sim -\sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(1-\alpha n)} \left(\frac{\tau}{t}\right)^{\alpha n}$$

$$\frac{d\phi(t)}{dt} = {}_0D_t^{1-\alpha}\phi(t) \equiv \frac{d}{dt}\int_0^t \frac{(t-t')^{\alpha-1}}{\Gamma(\alpha)} dt'$$

Agricultural Rheology

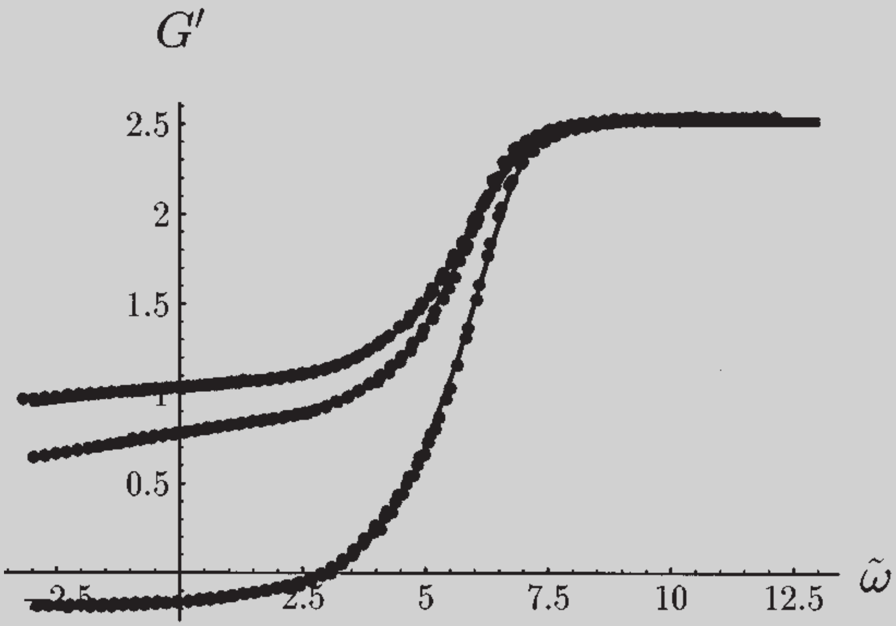
G. W. SCOTT BLAIR
AND MARKUS REINER

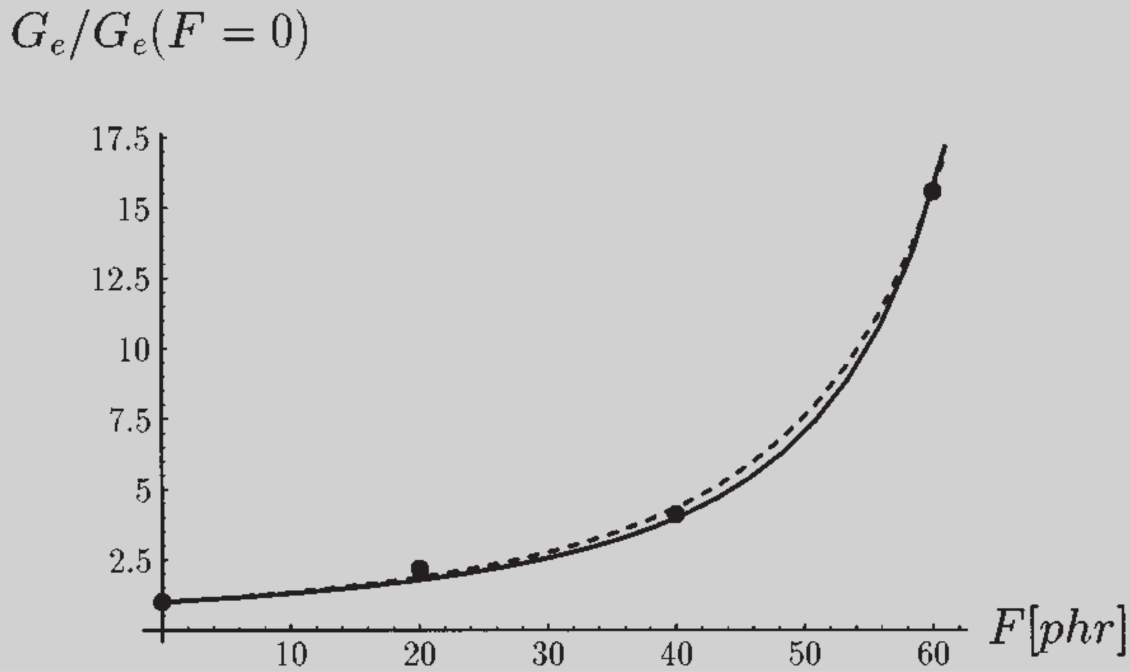

COLIN McKENZIE BAKER
HIS WIFE
JANET FOSTER
MARION McKENZIE BAKER
PAGE MERCER BAKER
ALICE WALKER BAKER

Moduli of carbon-black filled rubbers

Fractional Zener model of rheology:

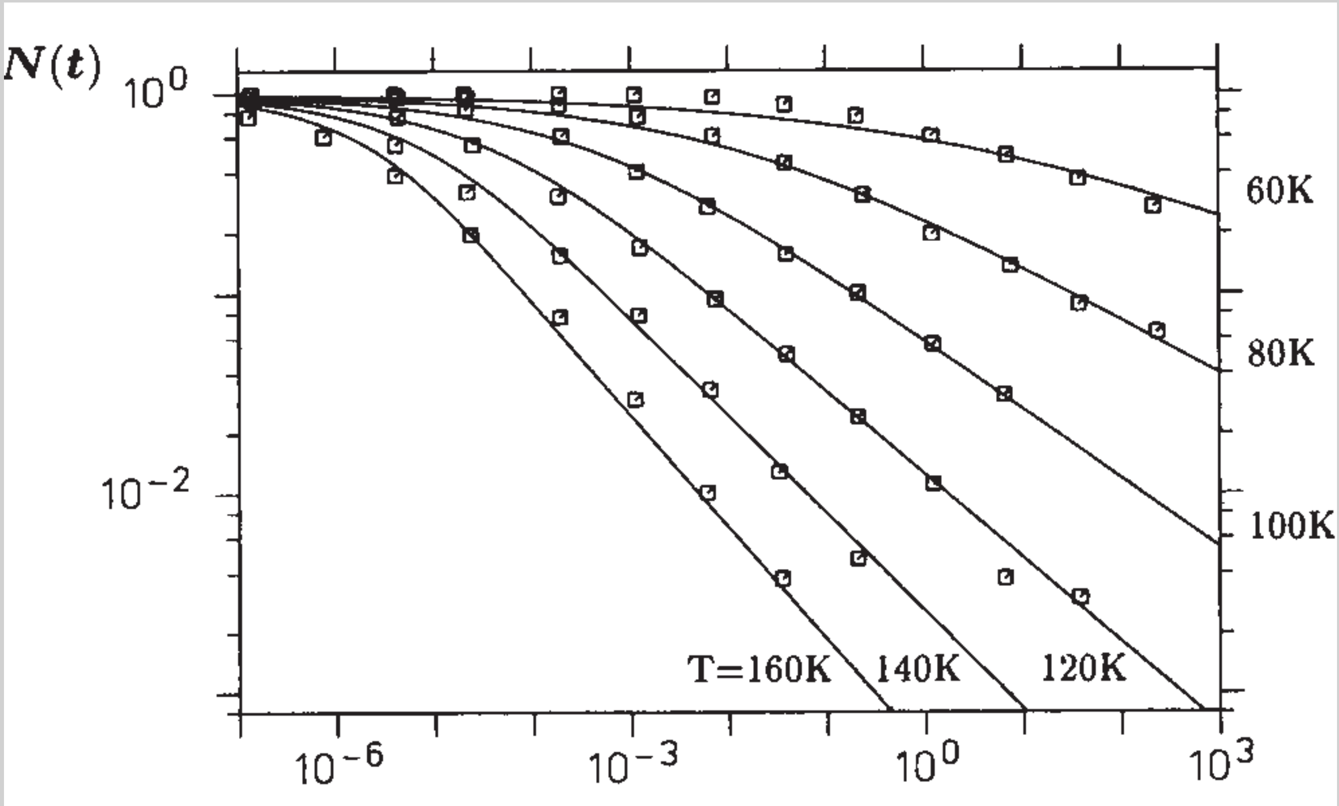
$$\sigma(t) + \tau_0^q \, {}_0D_t^q\Big(\sigma(t) - \sigma_0\Big) = \Big(G_m + G_e + G_\infty\Big) \, {}_0D_t^q \tau_0^q \Big(\epsilon(t) - \epsilon_0\Big) + G_e \tau_0^{q-\mu} \, {}_0D_t^{q-\mu} \epsilon(t) + G_\infty \epsilon(t)$$





RM, W Schick, HG Kilian & TF Nonnenmacher, J Chem Phys (1995)

Rebinding of CO to myoglobin

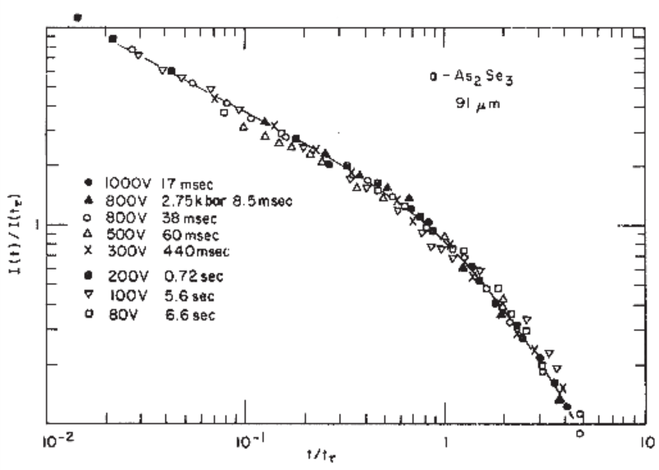


WG Glöckle & TF Nonnenmacher, Biophys J (1995)

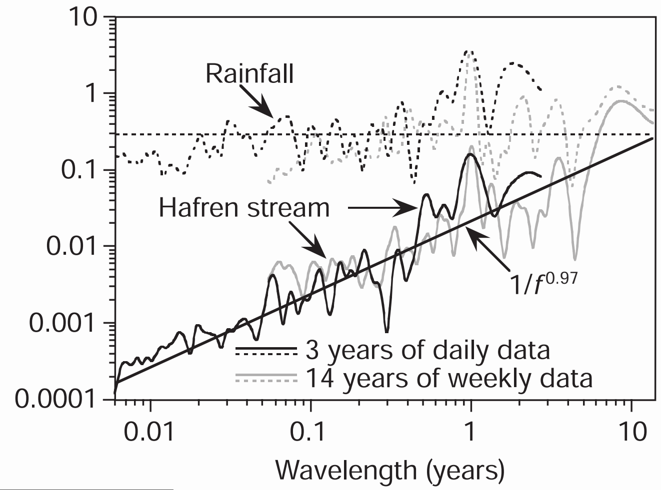
Anomalous diffusion occurs from micro to macro scales

Glasses/disordered systems

Scher & Montroll, PRB (1975)



Tracer dispersion in groundwater



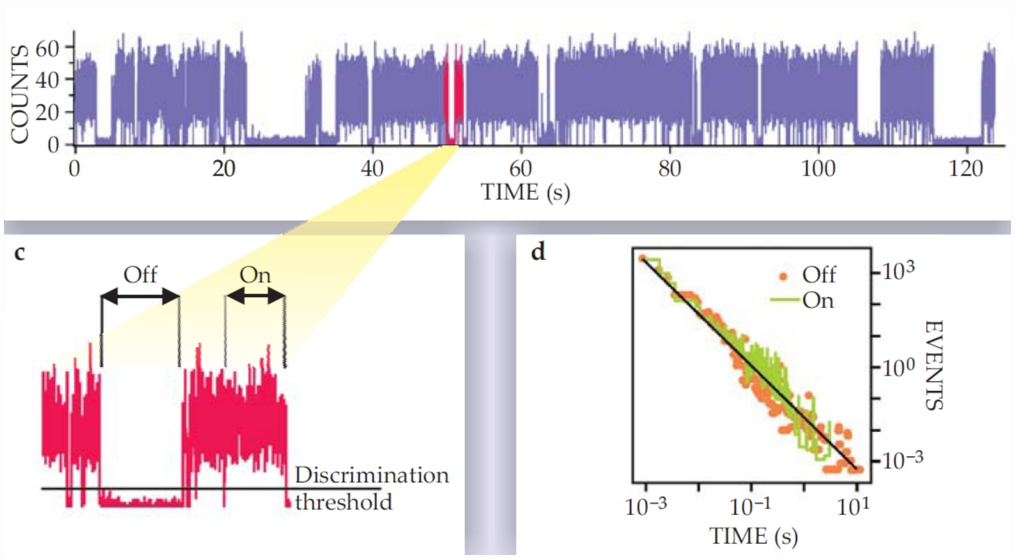
Scher et al, GRL (2002)

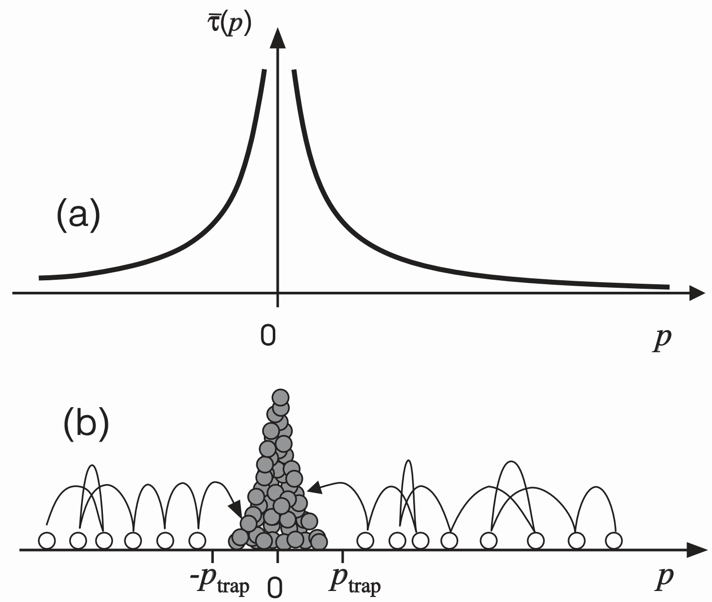
Blinking quantum dots

$$\langle \mathbf{r}^2(t) \rangle \simeq t^\alpha$$

Subrecoil laser cooling

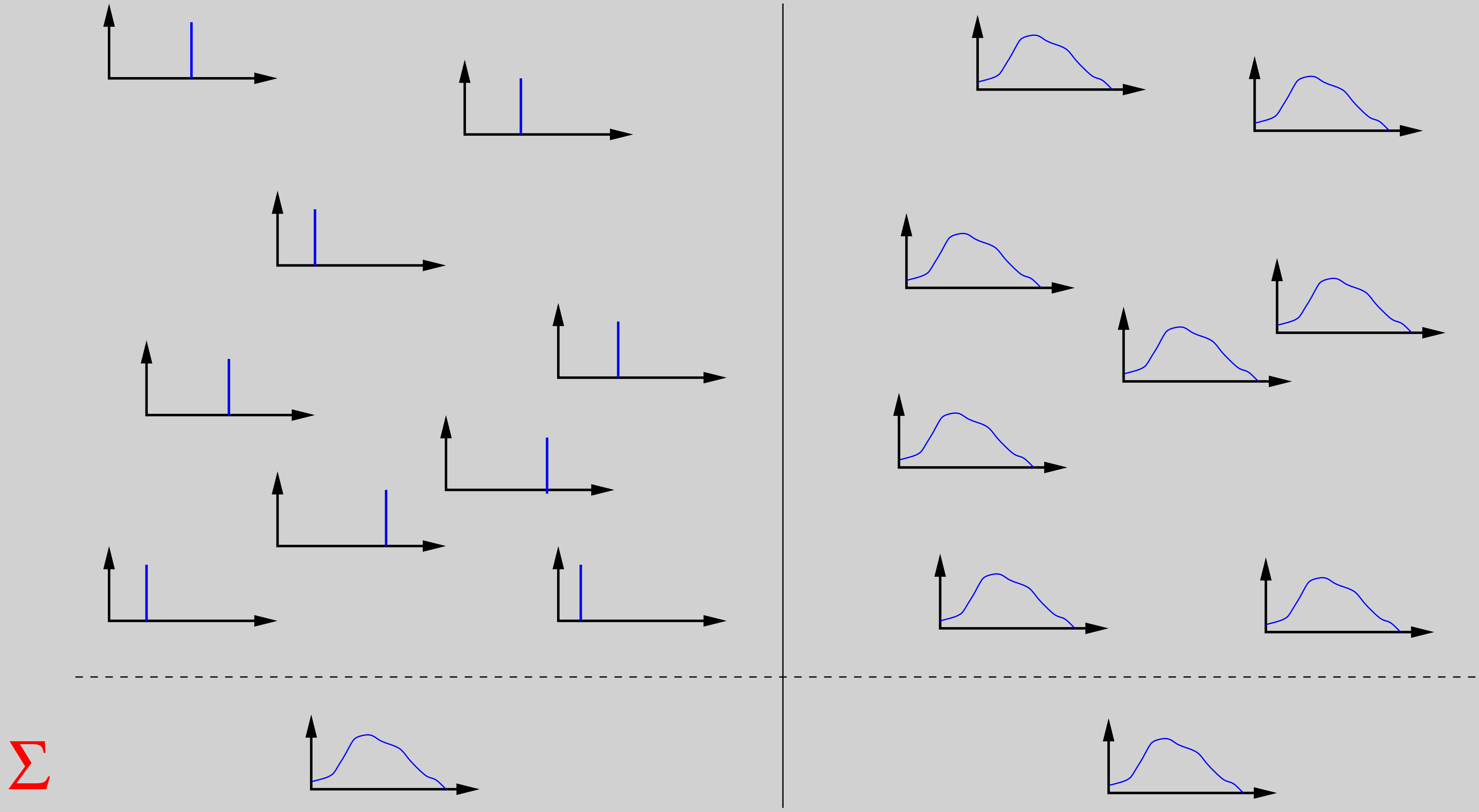
Stefani et al, Phys Today (2009)





Bardou, Chaotic Dynamics etc (2005)

Single molecule experiments vs ensemble average



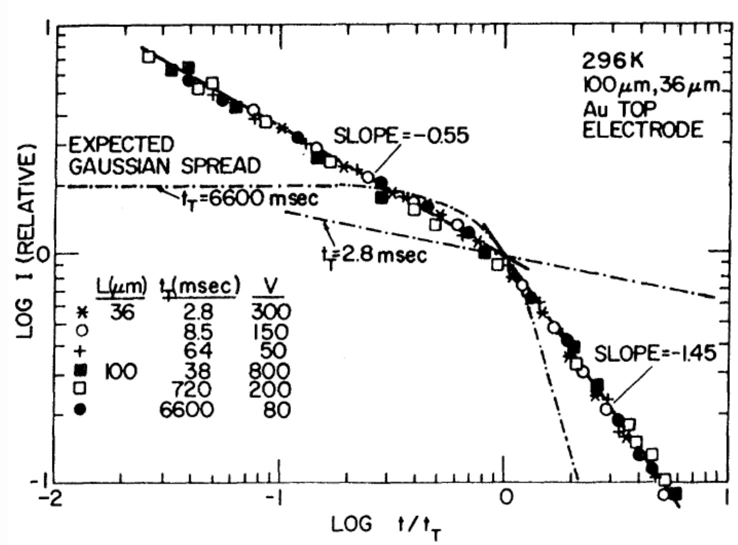
Charge carrier diffusion in amorphous semiconductors

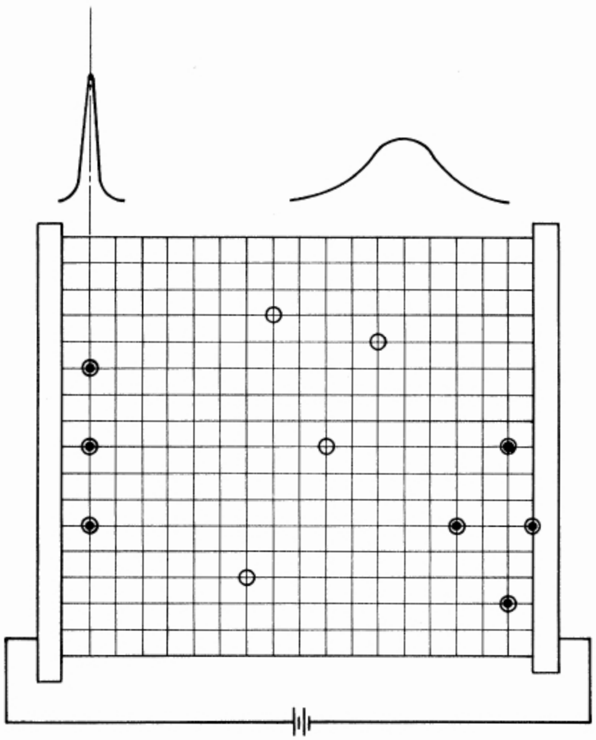
Waiting time distribution

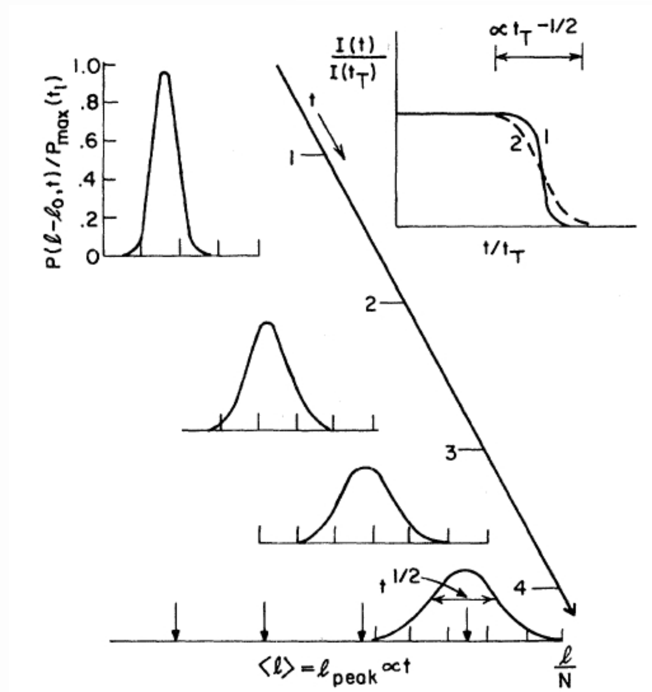
$$\psi(t) \simeq \frac{\tau^{\alpha}}{t^{1+\alpha}} \Leftrightarrow \psi(u) = \exp\left(-[u\tau]^{\alpha}\right)$$

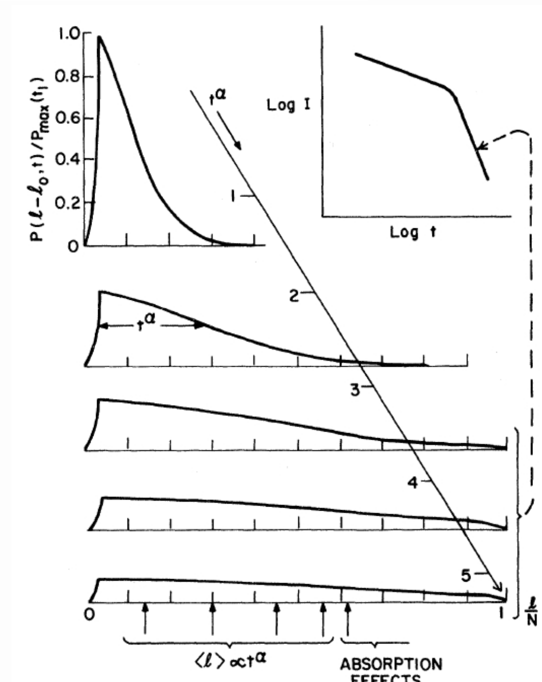
In the bias of the electrical field:

$$\frac{\sqrt{\langle(\Delta x)^2\rangle}}{\langle x(t)\rangle} \simeq t^0 \quad \leftrightsquigarrow \quad \underbrace{\frac{\sqrt{\langle(\Delta x)^2\rangle}}{\langle x(t)\rangle} \simeq t^{-1/2}}_{\alpha=1}$$



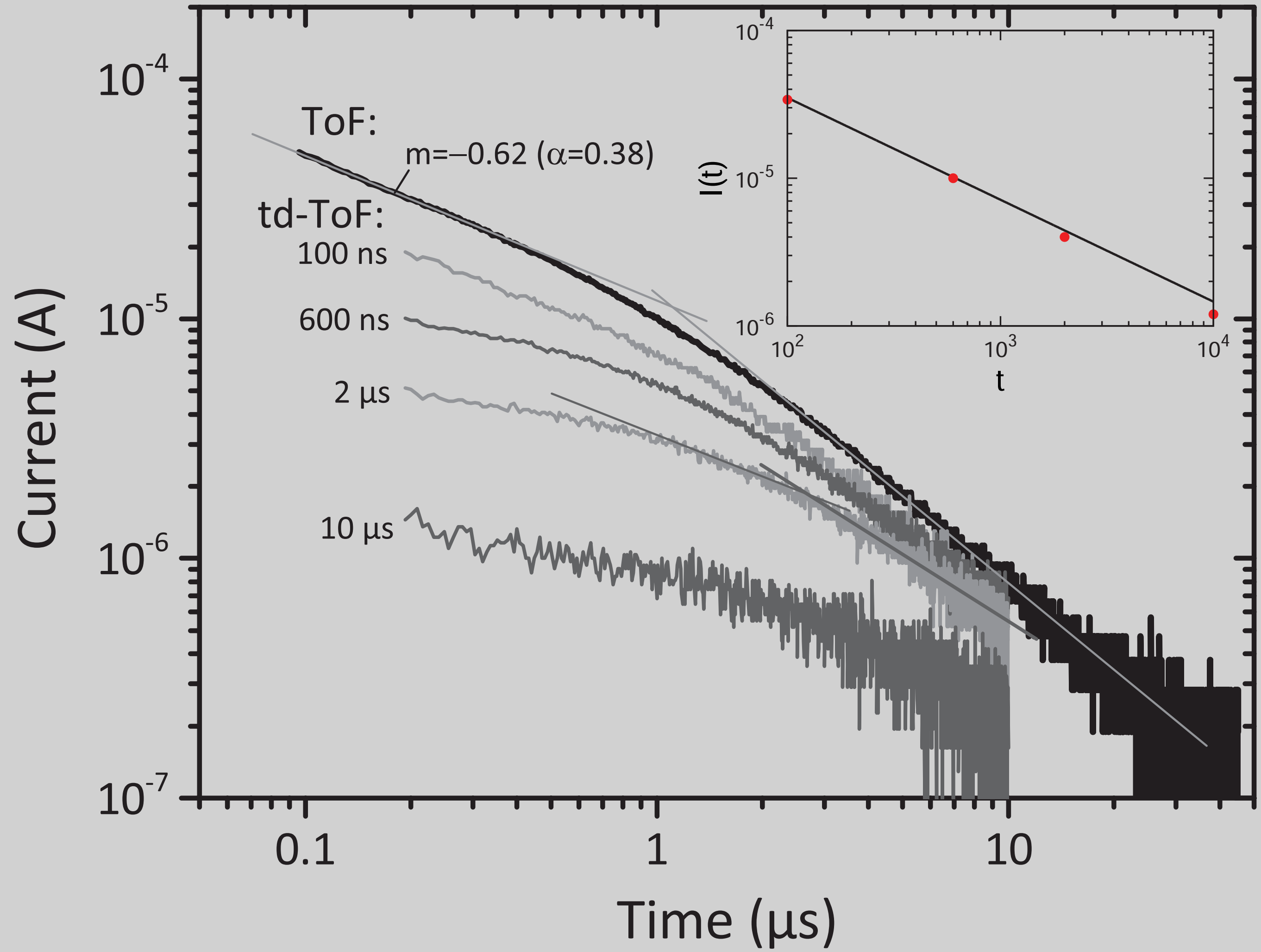






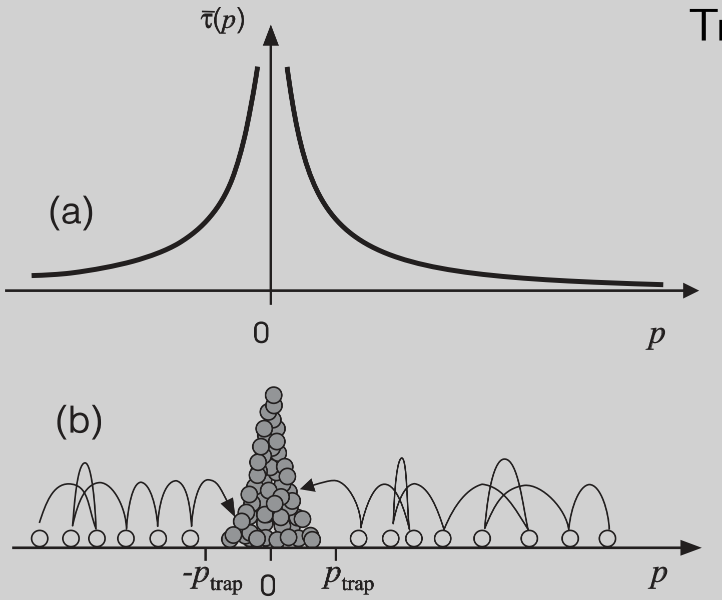
H Scher & E Montroll, PRB (1975)

Ageing charge carrier motion in polymeric semiconductors



M Schubert, . . . & D Neher, Phys Rev B (2013); H Krüsemann, A Godec & RM, PRE (2014), JPA (2015), TiPM (2016)

Power-law waiting time dynamics in subrecoil laser cooling



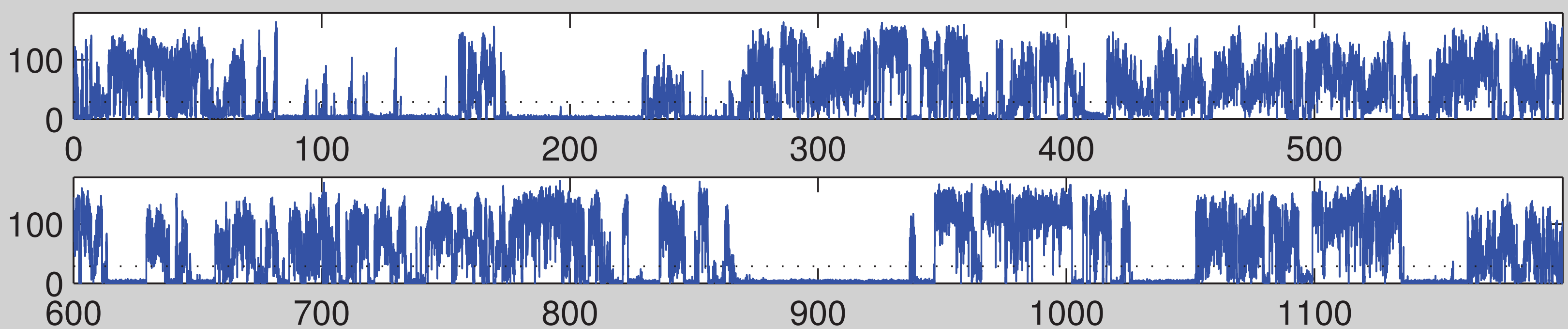
Trapping time PDF of particles near zero momentum:

$$\psi(\tau) \simeq \tau^{-3/2}$$

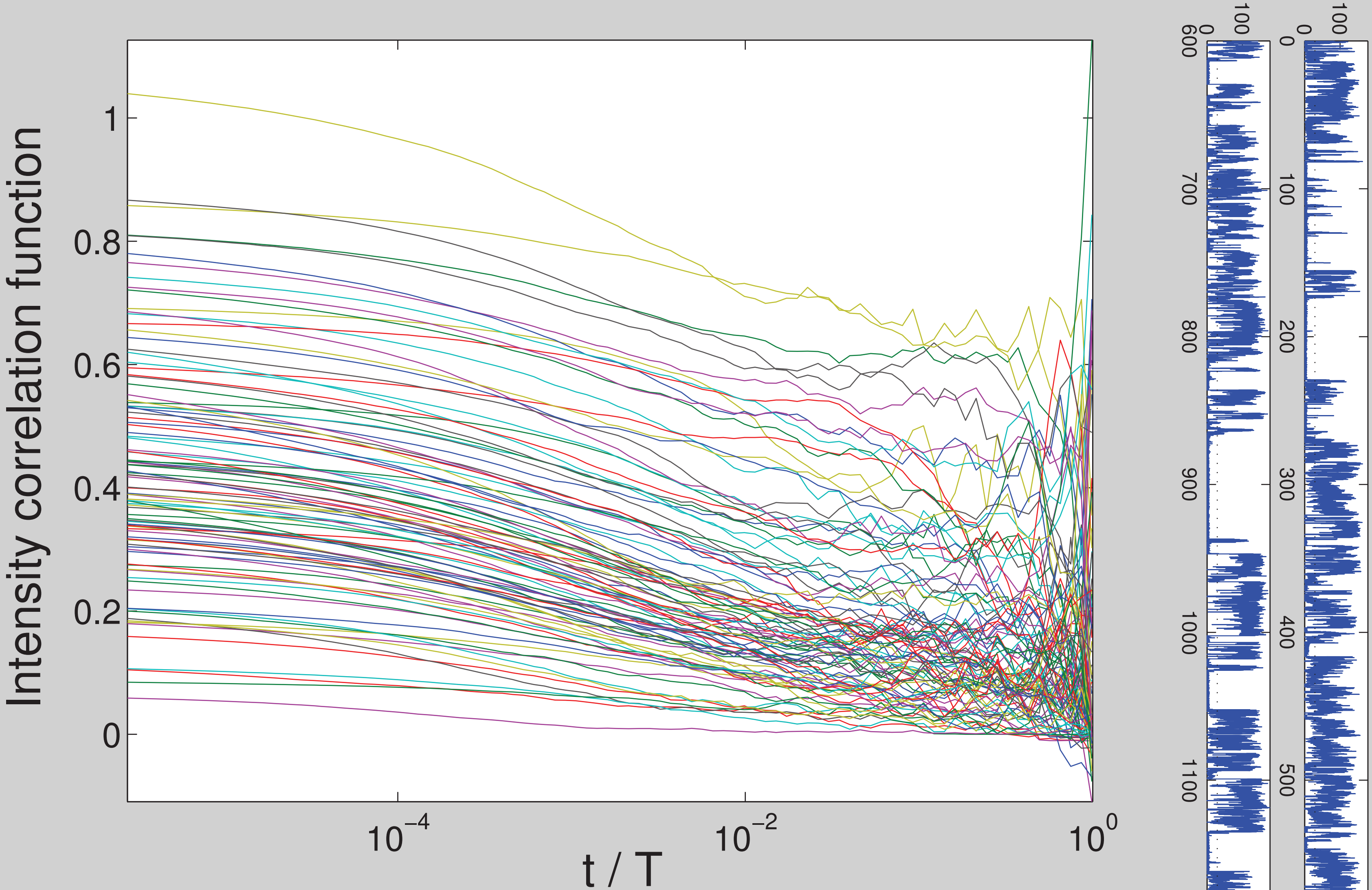
F Bardou, JP Bouchaud, O Emile, A Aspect & C Cohen-Tannoudji, PRL (1994)

S Schaufler, WP Schleich, VP Yakovlev, PRL (1999)

Power-law switching times in quantum dots



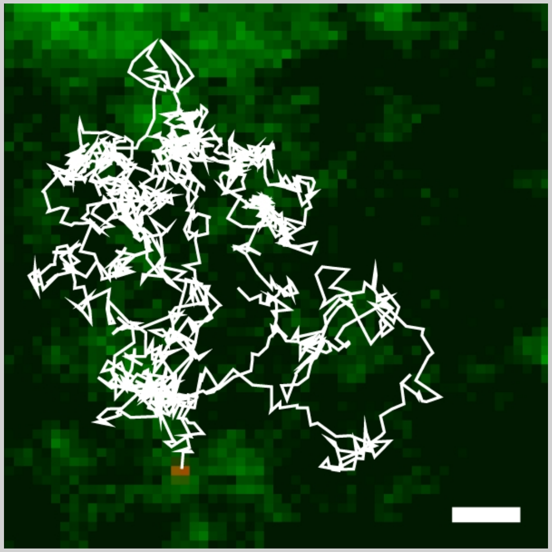
Blinking dynamics in quantum dots: amplitude scatter

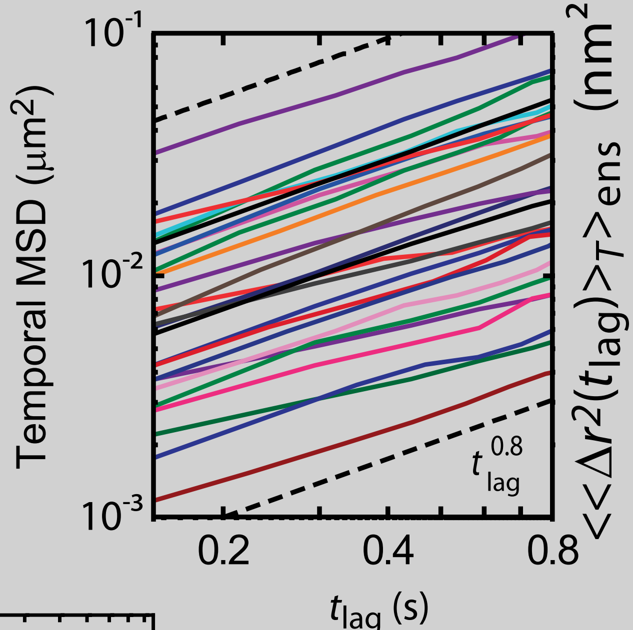


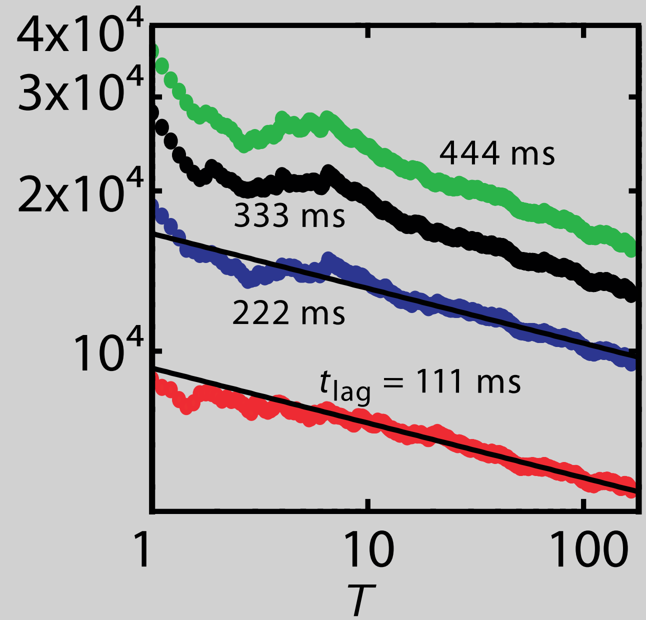
On/off time statistics indeed follow power-laws with $\alpha \approx 0.5$

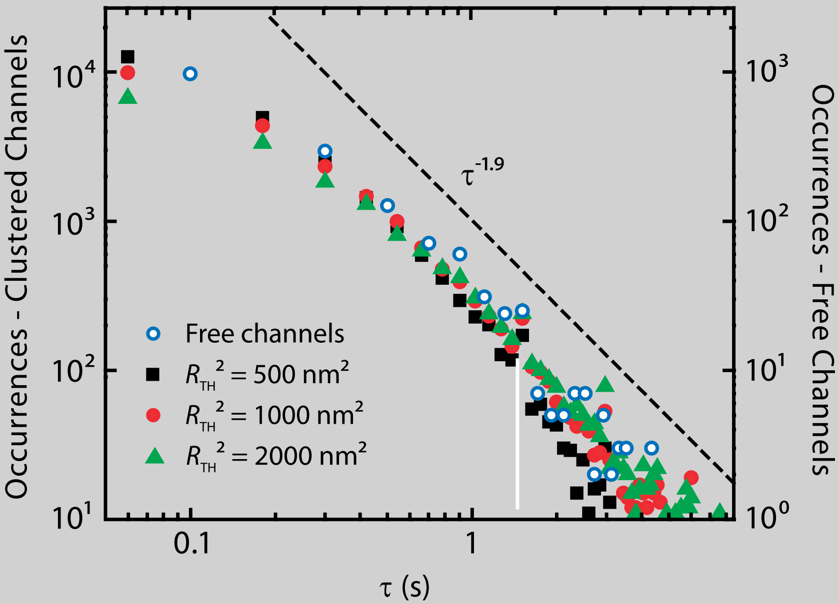
FD Stephani, JD Hoogenboom & E Barkai, Phys Today (2009)

CTRW-like motion of Ka channels in plasma membrane









$\psi(\tau) \simeq \tau^{-1-\alpha}$ scale free

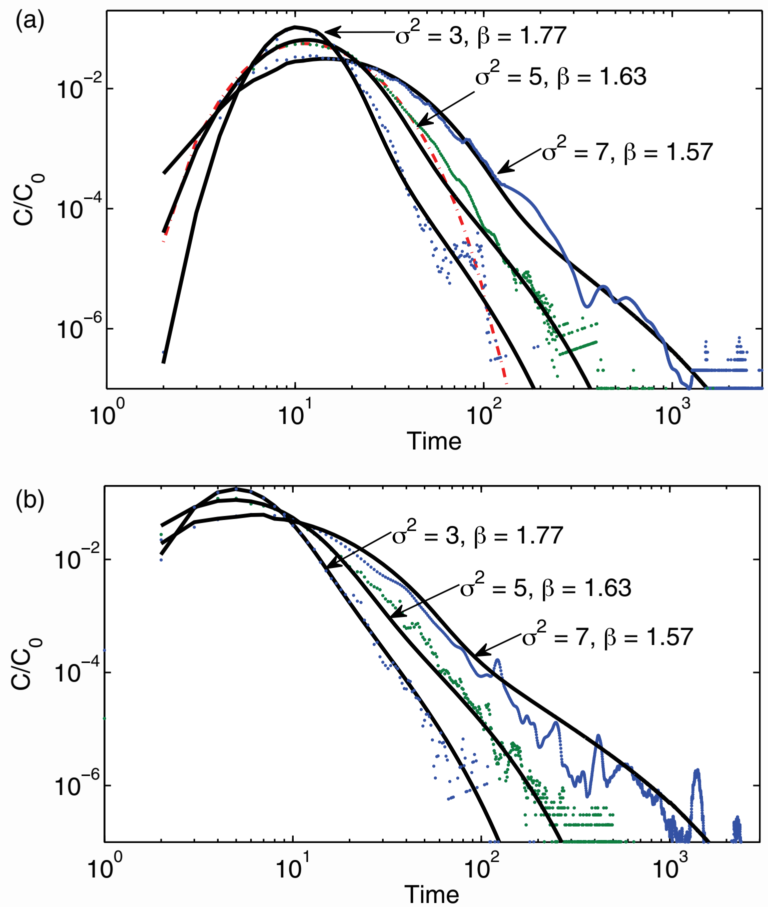
$\overline{\delta^2(\Delta)}$ apparently random

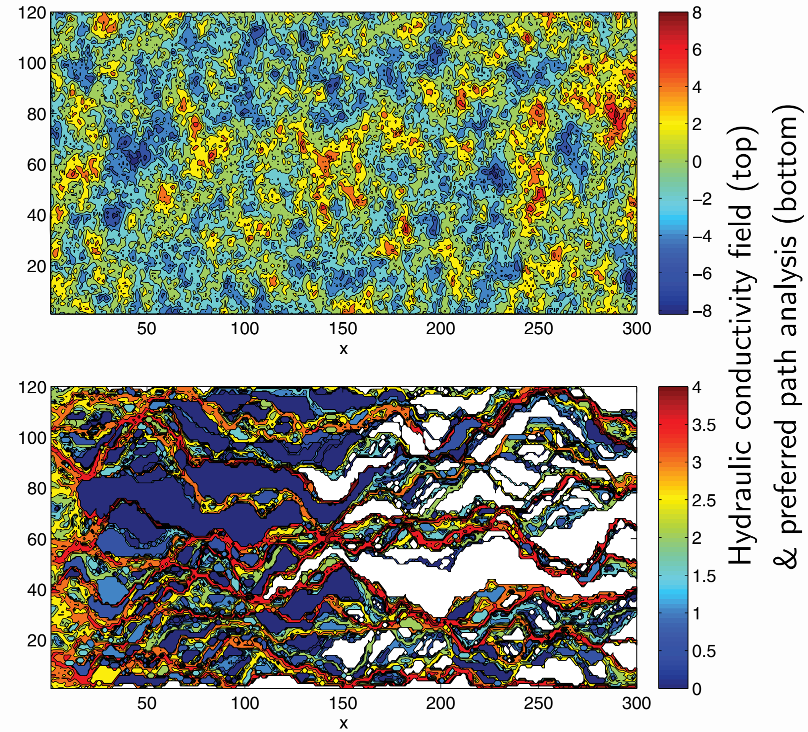
$$\Delta/T^{1-\alpha} \simeq \overline{\delta^2(\Delta)} \neq \langle \mathbf{r}^2(\Delta) \rangle \simeq \Delta^\alpha$$

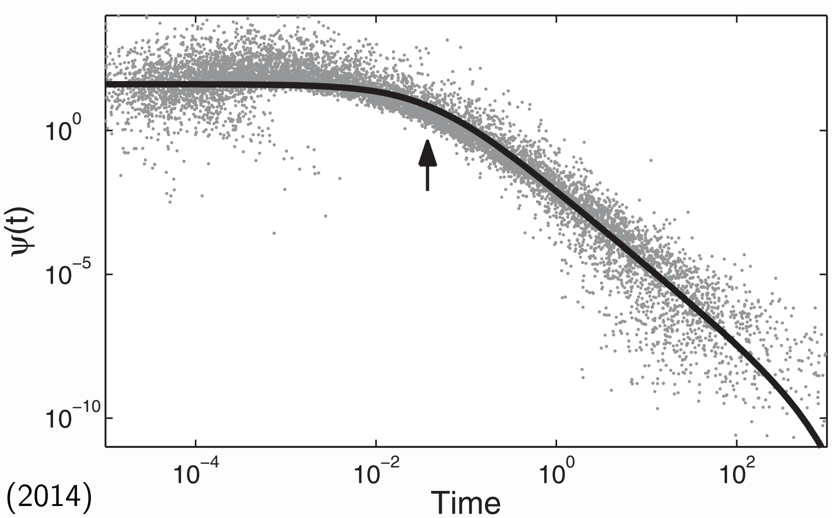
$$P(\mathbf{r},t) \simeq \exp\left(-\beta r^{1/[1-\alpha/2]}\right)$$

AV Weigel, B Simon, MM Tamkun & D Krapf, PNAS (2011)

Reconstruction of waiting time PDF from hydraulic field







Reconstructed $\psi(t)$ fitted by parameters from the corresponding breakthrough curves

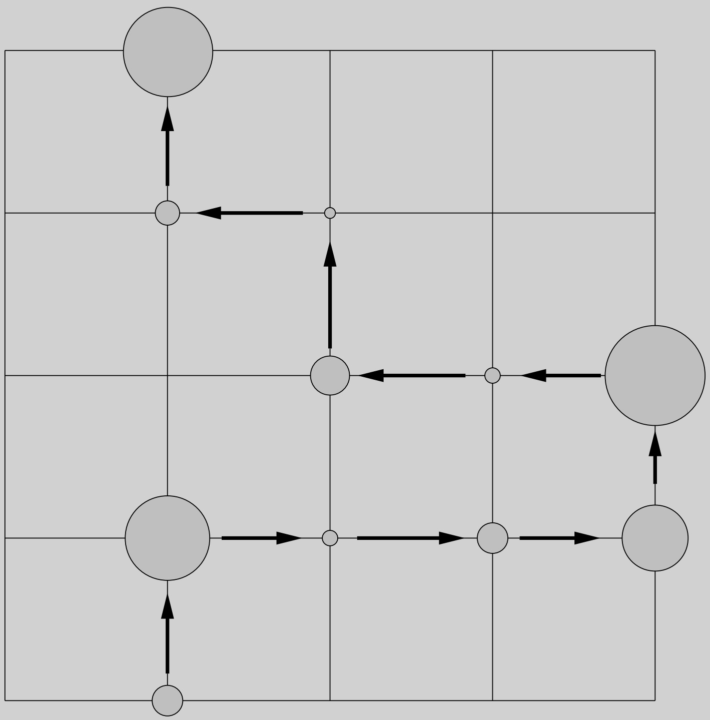
Y Edery, A Guadagnini, H Scher & B Berkowitz, Wat Res Res (2014)

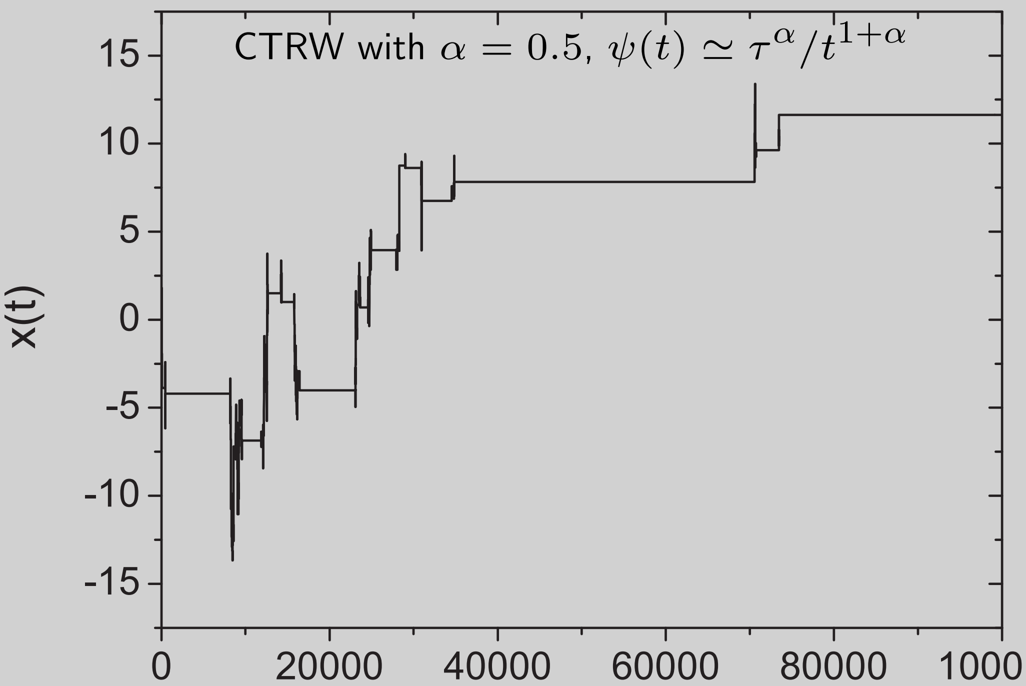
Y Edery, A Guadagnini, H Scher & B Berkowitz, Wat Res Res (2014)

The Continuous Time Random Walk (CTRW) model

Waiting time distribution $\psi(t)$; characteristic waiting time $\langle t\rangle = \int_0^\infty t\psi(t)dt \to \infty$

Jump length distribution $\lambda(x)$; jump length variance $\langle \delta x^2\rangle = \int_{-\infty}^\infty x^2\lambda(x)dx < \infty$





Montroll & Weiss, J Stat Phys (1969); Scher & Montroll, Phys Rev B (1975)

Scher-Montroll equation

$$\eta(x,t) = \delta(t)\delta(x) + \int_0^t dt' \int_{-\infty}^{\infty} dx' \eta(x',t')\lambda(x-x')\psi(t-t')$$

$$\eta(k,u) = 1 + \eta(k,u)\lambda(k)\psi(u)$$

$$P(x,t) = \int_0^t dt' \eta(x,t')\Psi(t-t') \;\therefore\; \Psi(t) = 1 - \int_0^t dt' \psi(t')$$

$$P(k,u) = \eta(k,u)\Psi(u) = \eta(k,u)\frac{1-\psi(u)}{u}$$

$$\boxed{P(k,u) = \frac{1-\psi(u)}{u}\frac{1}{1-\lambda(k)\psi(u)} \;\therefore\; \psi(u) \sim 1-(u\tau)^{\alpha},\;\; \lambda(k) \sim 1-(\sigma|k|)^{\mu}}$$

J Klafter, A Blumen & MF Shlesinger, PRA (1987)

A primer on the fractional Fokker-Planck equation

Waiting time and jump length densities:

$$\psi(t) \simeq \frac{\tau^\alpha}{t^{1+\alpha}},\ 0<\alpha<1;\ \ \Lambda(x,x') = \lambda(x-x')\Big(A(x')\Theta(x-x')+B(x')\Theta(x'-x)\Big)$$

Continuum limit: fractional Fokker-Planck equation:

$$\frac{\partial}{\partial t}P(x,t) = {}_0D_t^{1-\alpha}\left(\frac{\partial}{\partial x}\frac{V'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2}\right)P(x,t) \quad \therefore \quad K_\alpha = \frac{k_B T}{m\eta_\alpha}$$

Riemann-Liouville operator: Subordination:

$${}_0D_t^{1-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{f(t')}{(t-t')^{1-\alpha}}dt' \ \ \therefore\ \ P(x,t) = \int_0^\infty \mathscr{E}_\alpha(s,t)P_M(x,s)ds$$

Generalised Laplace kernel ($s^* \equiv \eta_\alpha s/\eta_1$):

$$\mathscr{E}_\alpha(s,t) = \mathscr{L}^{-1}\left\{\frac{\eta_\alpha}{\eta_1 u^{1-\alpha}}\exp\left(-u^\alpha s^*\right)\right\} = \frac{1}{\alpha s}H_{1,1}^{1,0}\left[\frac{(s^*)^{1/\alpha}}{t}\middle| \begin{array}{l}(1,1)\\(1,1/\alpha)\end{array}\right]$$

Mittag-Leffler form of the mode relaxation:

$$T_n(t) = T_0 E_\alpha\Big(-\lambda_n t^\alpha\Big) = \sum \frac{(-\lambda_n t^\alpha)^n}{\Gamma(1+\alpha n)} \sim \left\{\begin{array}{l} 1-\lambda_n t^\alpha/\Gamma(1+\alpha) \\ 1/\left(\Gamma(1-\alpha)\lambda_n t^\alpha\right)\end{array}\right.$$

RM, E Barkai & J Klafter, PRL (1999), EPL (1999); RM & J Klafter, Phys Rep (2000); J Phys A Rev (2004)

Fractional diffusion equation

Time-fractional diffusion equation:

$$\frac{\partial}{\partial t}P(x,t) = {}_0D_t^{1-\alpha}K_\alpha\frac{\partial^2}{\partial x^2}P(x,t)$$

Equivalent version:

$${}_0D_t^{\alpha}P(x,t) - \frac{1}{\Gamma(1-\alpha)}t^{-\alpha} = \frac{\partial^2}{\partial x^2}P(x,t)$$

Initial value also needed for consistency: after integration $\int \cdot dx$,

lhs: ${}_0D_t^{\alpha}1 - \frac{1}{\Gamma(1-\alpha)}t^{-\alpha} = 0$

rhs: $K_\alpha[\partial P(x,t)/\partial x]_{-\infty}^{\infty} = 0$

Alternatively, use Caputo operator

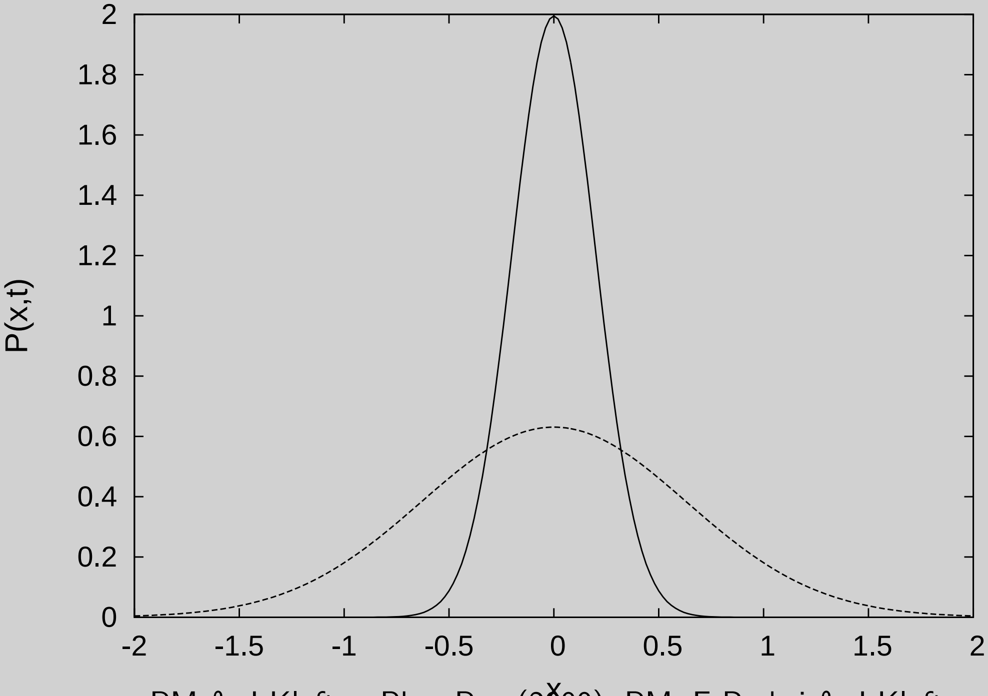
RM & J Klafter, Phys Rep (2000); RM, E Barkai & J Klafter, PRL (1999)

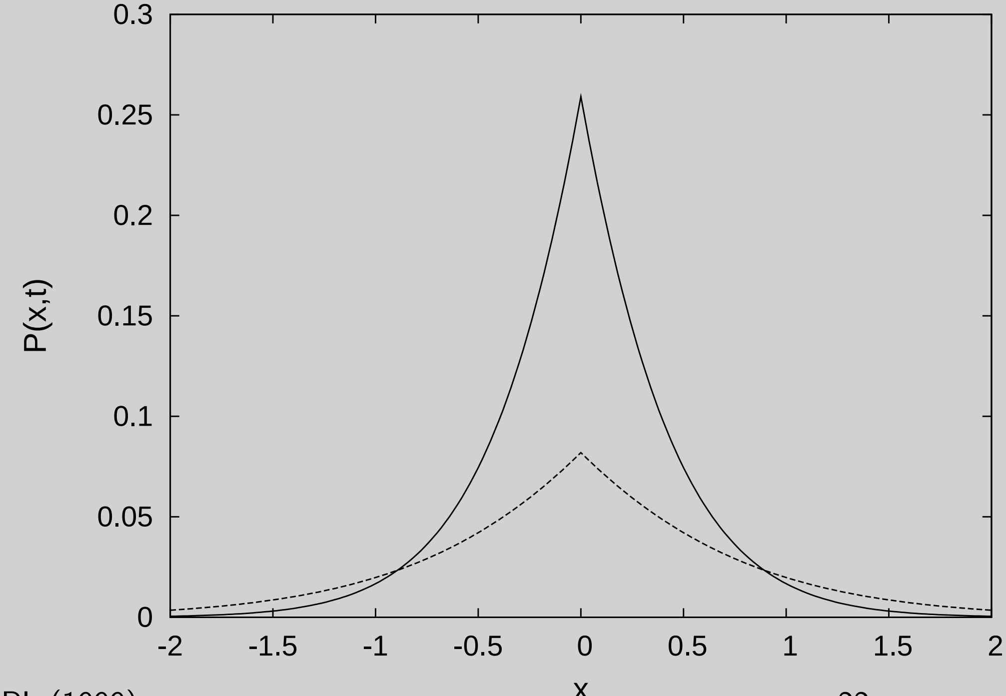
Solution of the fractional diffusion equation

$$P(x,t) = \frac{1}{\sqrt{4\pi K_\alpha t^\alpha}} H^{2,0}_{1,2}\left[\frac{x^2}{4K_\alpha t^\alpha}\middle|\begin{array}{l}\left(1-\frac{\alpha}{2},\alpha\right)\\(0,1),\left(\frac{1}{2},1\right)\end{array}\right]$$

$$P(x,t) \sim \frac{1}{\sqrt{4\pi K_\alpha t^\alpha}}\sqrt{\frac{1}{2-\alpha}}\left(\frac{2}{\alpha}\right)^{(1-\alpha)/(2-\alpha)}\left(\frac{|x|}{\sqrt{K_\alpha t^\alpha}}\right)^{-(1-\alpha)/(2-\alpha)}$$

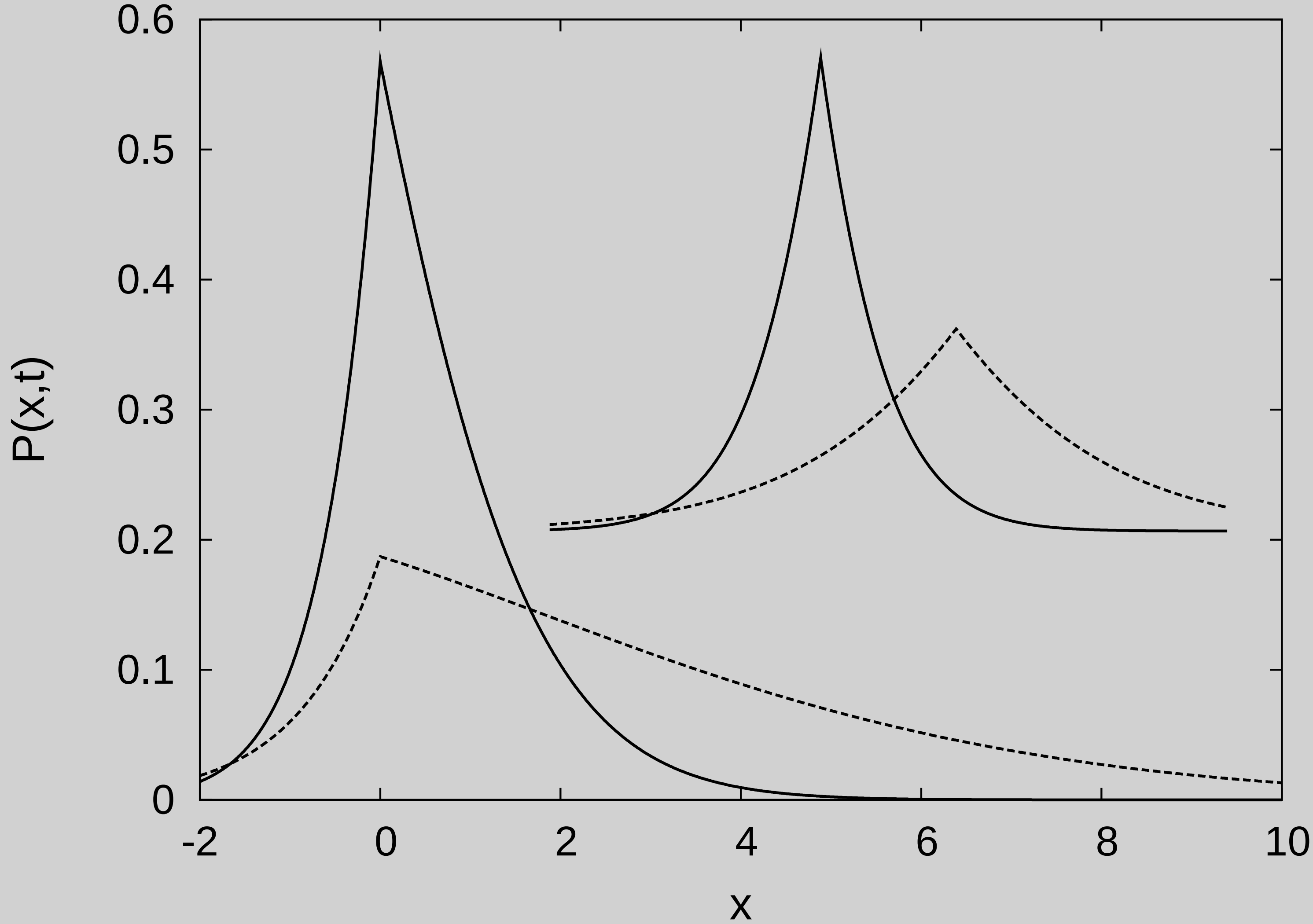
$$\times \exp\left(-\frac{2-\alpha}{2}\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left[\frac{|x|}{\sqrt{K_\alpha t^\alpha}}\right]^{1/(1-\alpha/2)}\right)$$





RM & J Klafter, Phys Rep (2000); RM, E Barkai & J Klafter, PRL (1999)

Fractional diffusion-advection



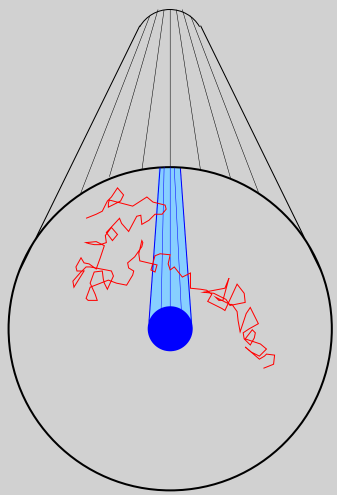
Lack of time scale causes weak ergodicity breaking (WEB)

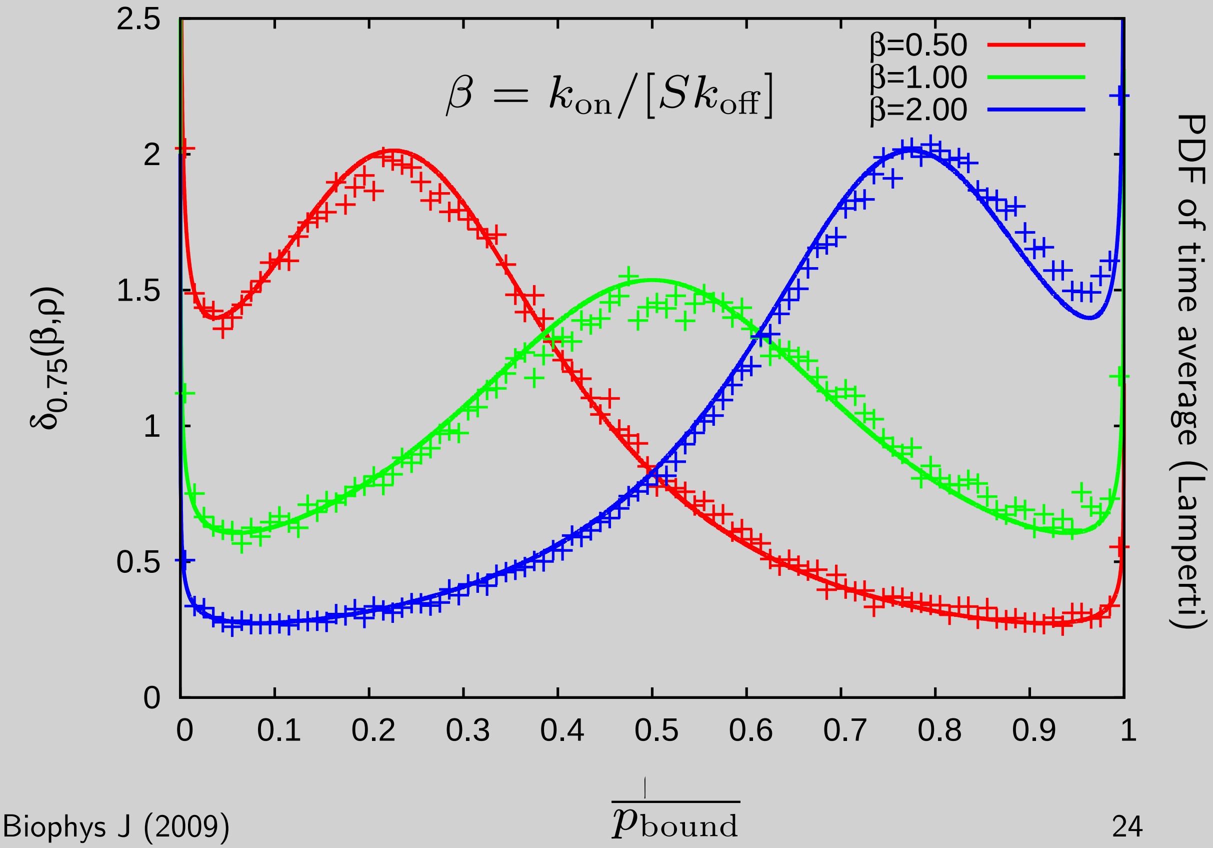
Unbinding time PDF to bulk, rebinding time PDF to cylinder for $\psi(t) \simeq t^{-1-\alpha}$:

$$\wp_{\rm unb}(t) \sim \frac{\kappa^{1-\alpha}}{\kappa_{\rm off} t^{1+\alpha}};\quad \wp_{\rm reb}(t) \sim \frac{S}{k_{\rm on} t^{1+\alpha}},\quad S = \pi(R_2^2 - R_1^2)$$

In this system time averages are random & different from ensemble averages: WEB

$$\overline{p_{\rm bound}} \equiv \lim_{t\to\infty} \frac{t_{\rm bound}}{t} \neq \langle p_{\rm bound}\rangle \equiv \frac{k_{\rm on}}{k_{\rm on} + S k_{\rm off}/k_{\rm on}}$$





Extracting information from single Brownian trajectories

Ensemble averaged MSD for normal diffusion (on average # jumps $\sim$ elapsed time t):

$$\left\langle \mathbf{r}^2(t) \right\rangle = \int \mathbf{r}^2 P(\mathbf{r}, t) d\mathbf{r} = 2dK_1 t \quad \left(= \langle \delta\mathbf{r}^2 \rangle \frac{t}{\tau}, \quad K_1 = \frac{\langle \delta\mathbf{r}^2 \rangle}{2d\tau} \right)$$

Single particle trajectory $\mathbf{r}(t)$, $t \in [0, T]$:

$$\overline{\delta^2(t)} = \frac{1}{T-t} \int_0^{T-t} \Big[\mathbf{r}(t'+t) - \mathbf{r}(t') \Big]^2 dt' = \frac{1}{T-t} \int_0^{T-t} \langle \delta\mathbf{r}^2 \rangle \frac{t}{\tau} dt'$$

Single trajectory information equals ensemble information (*Boltzmann-Khinchin*):

$$\lim_{T\to\infty} \overline{\delta^2(t)} = 2dK_1 t = \left\langle \mathbf{r}^2(t) \right\rangle$$

Anomalous diffusion is not always ergodic (weak or strong violoation):

$$\left\langle \mathbf{r}^2(t) \right\rangle \simeq K_\alpha t^\alpha \neq \overline{\delta^2(t)} \simeq t/T^{1-\alpha}$$

WEB: signature of non-stationarity of process. SEB: discontinuity of phase space

E Barkai, Y Garini & RM, Phys Today (2012); Y He, S Burov, RM & E Barkai, PRL (2008); S Burov, RM & E Barkai, PNAS (2010)

Time averaged MSD & weak ergodicity breaking (WEB)

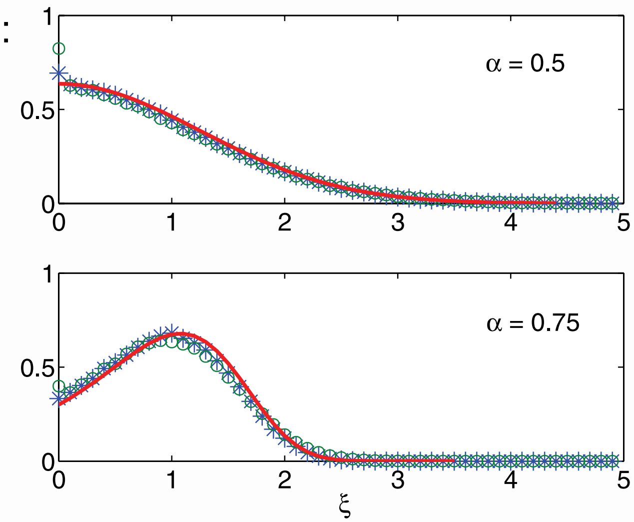
Time averaged MSD $\simeq \Delta$ is pseudo-Brownian and ageing ($\langle x^2(t)\rangle \simeq K_\alpha t^\alpha$):

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim \frac{1}{N}\sum_i^N \overline{\delta_i^2(\Delta)} \sim \frac{2dK_\alpha}{\Gamma(1+\alpha)}\frac{\Delta}{T^{1-\alpha}} \quad \therefore \quad K_\alpha \equiv \frac{\langle \delta \mathbf{r}^2\rangle}{2\tau^\alpha}$$

Amplitude distribution $\overline{\delta^2}$ of trajectories ($\xi \equiv \overline{\delta^2}/\langle\overline{\delta^2}\rangle$):

$$\phi_\alpha(\xi) \sim \frac{\Gamma^{1/\alpha}(1+\alpha)}{\alpha\xi^{1+1/\alpha}} L_\alpha^+\left(\frac{\Gamma^{1/\alpha}(1+\alpha)}{\xi^{1/\alpha}}\right)$$

$$\phi_{1/2}(\xi) = \frac{2}{\pi}\exp\left(-\frac{\xi^2}{\pi}\right); \quad \phi_1(\xi) = \delta(\xi-1)$$

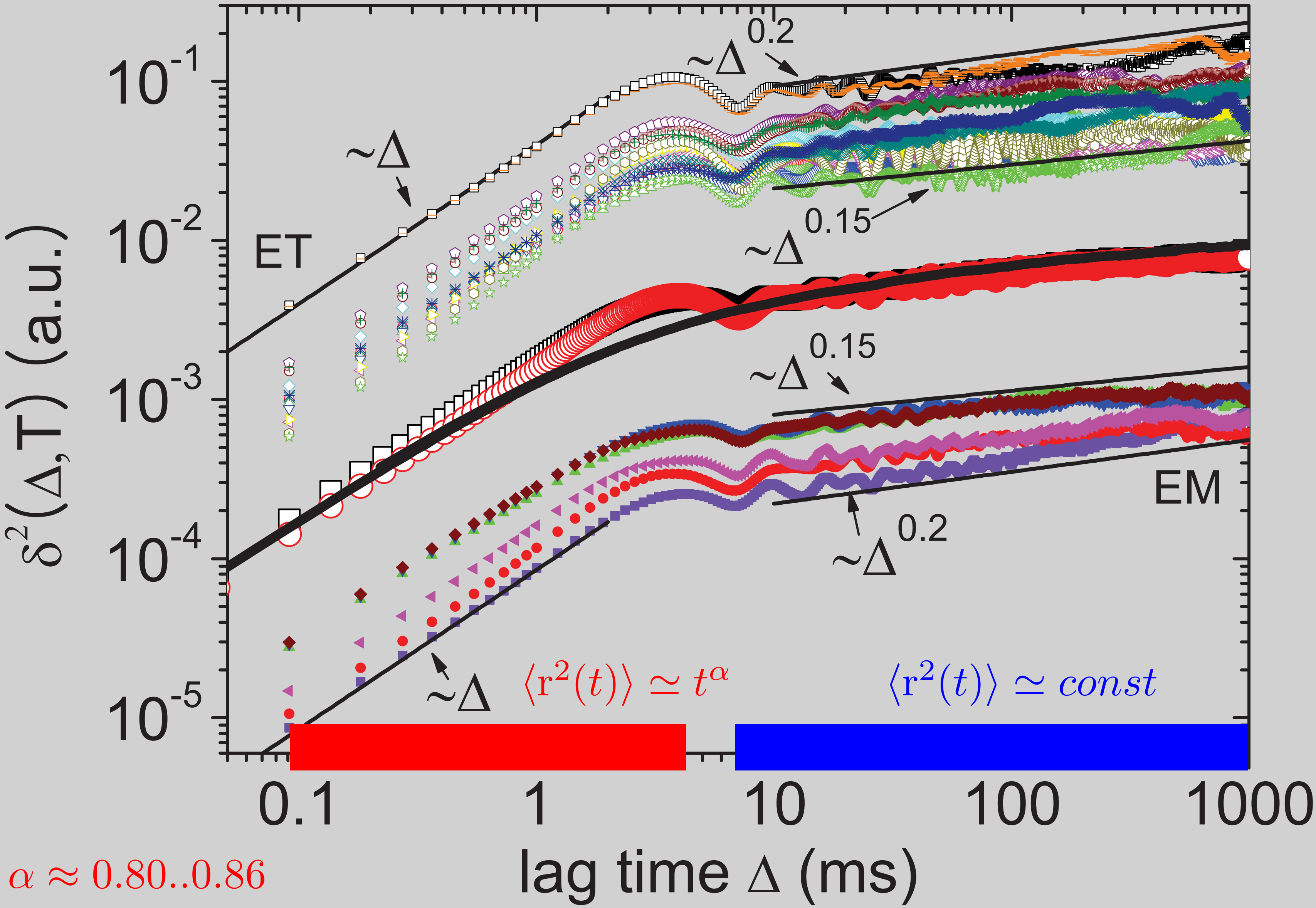


Confinement does not effect a plateau ($\langle x^2(t)\rangle \simeq \mathrm{const}(T)$):

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim \left(\left\langle x^2\right\rangle_B - \langle x\rangle_B^2\right)\frac{2\sin(\pi\alpha)}{(1-\alpha)\alpha\pi}\left(\frac{\Delta}{T}\right)^{1-\alpha}; \quad \frac{1}{(K_\alpha\lambda_1)^{1/\alpha}} \ll \Delta \ll T$$

Y He, S Burov, RM & E Barkai, PRL (2008); S Burov, RM, & E Barkai, PNAS (2010); JH Jeon . . . & RM, PRL (2011)

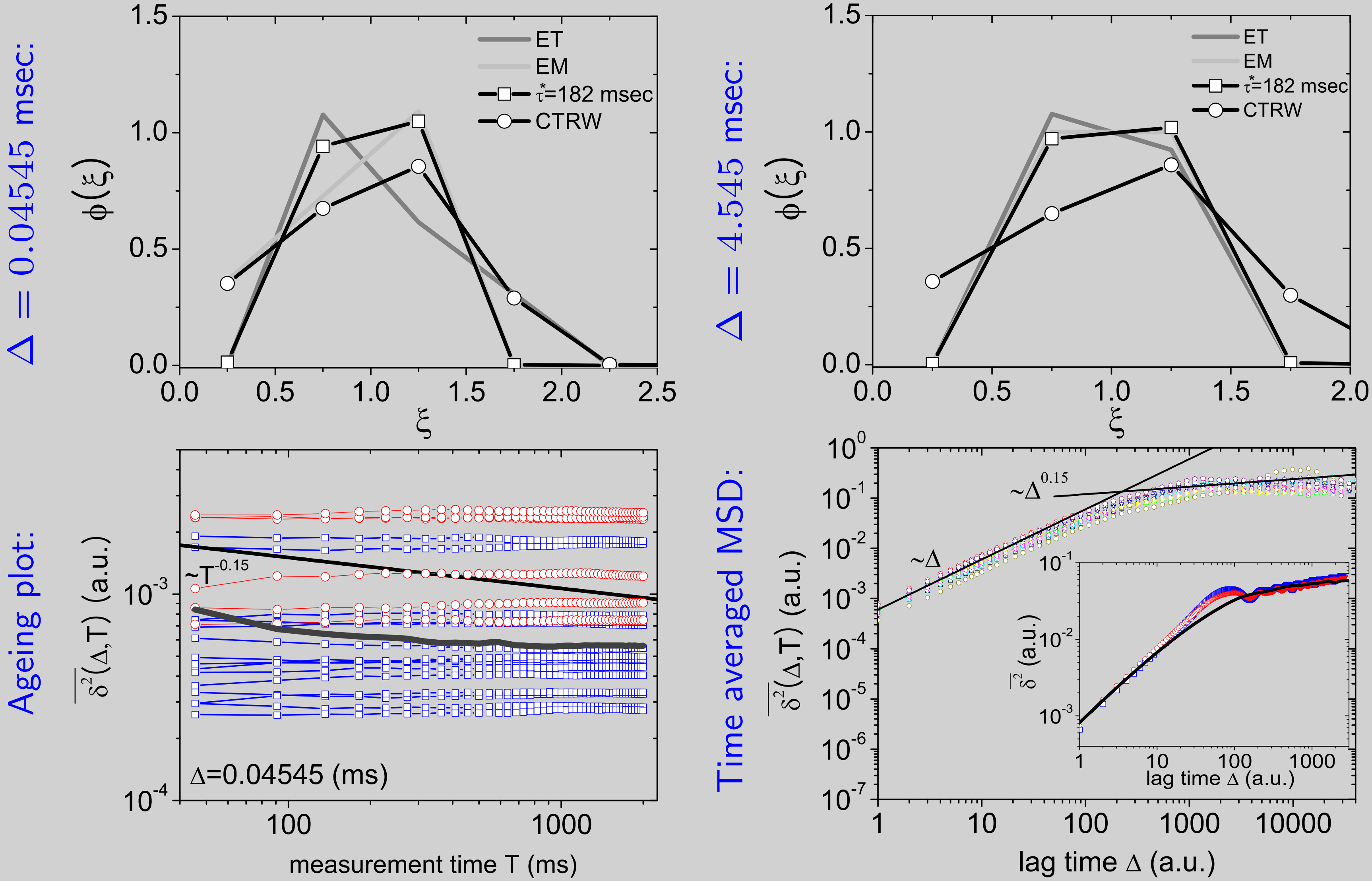
Granule subdiffusion in harmonic optical tweezer potential



$\alpha \approx 0.80..0.86$

J-H Jeon, V Tejedor, S Burov, E Barkai, C Selhuber-Unkel, K Berg-Sørensen, L Oddershede & RM, PRL (2011)

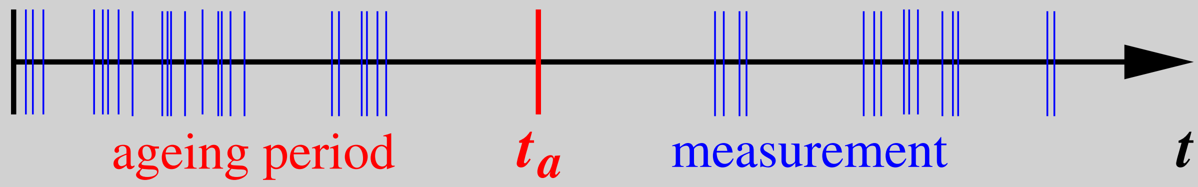
Further analysis of the lipid granule data



$$\psi(t) = \frac{d}{dt}\left[1 - \tau^{\alpha}(\tau + t)^{-\alpha}\exp\left(-\frac{t}{\tau^*}\right)\right] \quad \therefore \quad \tau^* \approx 182\text{msec}$$

J-H Jeon, V Tejedor, S Burov, E Barkai, C Selhuber-Unkel, K Berg-Sørensen, L Oddershede & RM, PRL (2011)

Ageing effects in single trajectory time averages

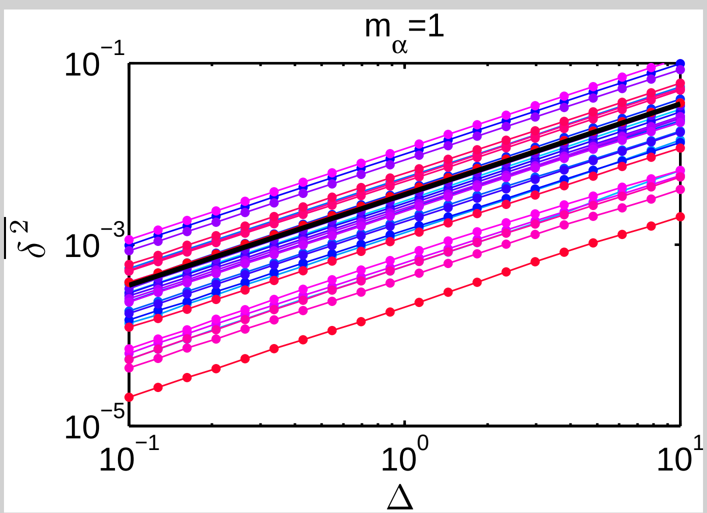


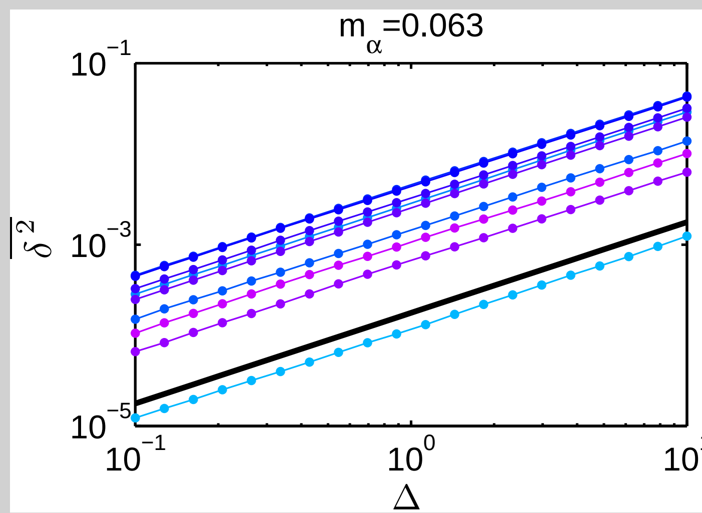
Ageing mean squared displacement $(\Lambda(z) = (1+z)^\alpha - z^\alpha)$

$$\left\langle\overline{\delta^2(\Delta)}\right\rangle_a = \frac{\Lambda_\alpha(t_a/T)}{\Gamma(1+\alpha)}\frac{g(\Delta)}{T^{1-\alpha}} \qquad \not\Leftrightarrow \qquad \langle x^2(t)\rangle_a \simeq \begin{cases} t^\alpha, & t_a \ll t \\ t_a^{\alpha-1}t, & t_a \gg t \end{cases}$$

Probability to make at least one step during $[t_a, t_a + T]$: *population splitting*

$$m_\alpha(T/t_a) \simeq (T/t_a)^{1-\alpha}, \quad T \ll t_a$$





J Schulz, E Barkai & RM, PRL (2013), PRX (2014)

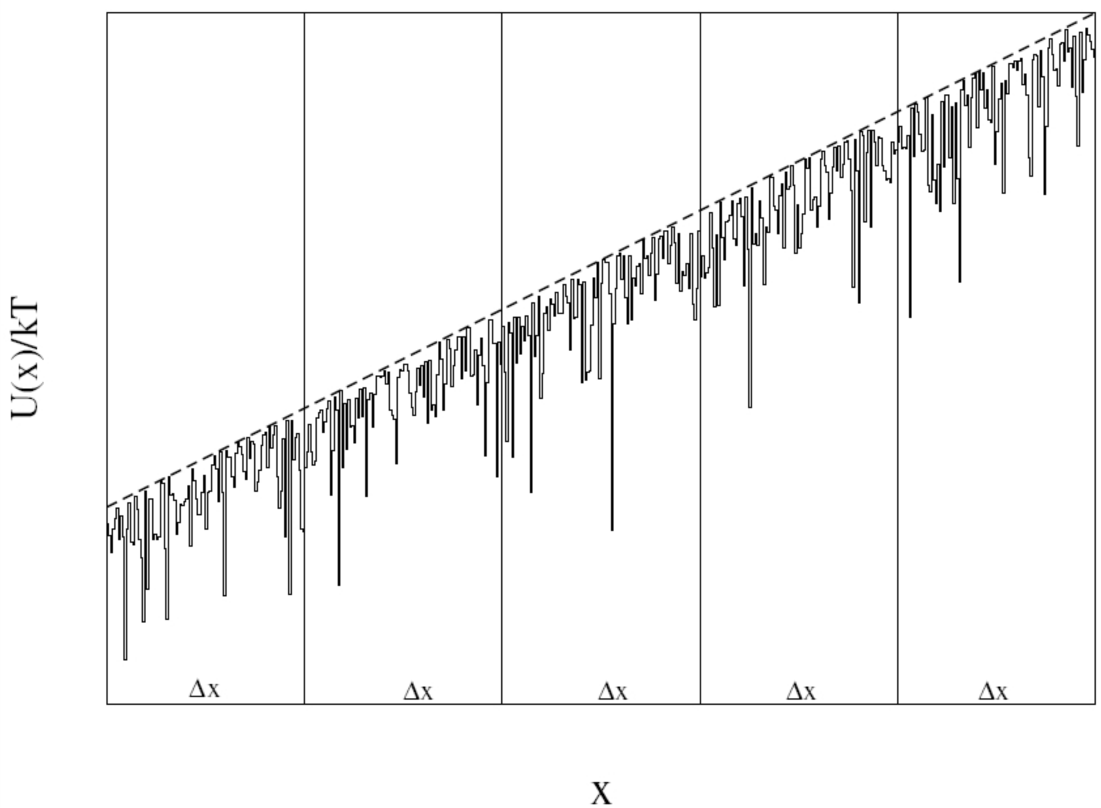
Quenched trap model for scale-free process

At each lattice site E_x is the depth of the trap. $\{E_x\}$ are iid random variables with PDF

$$\varrho(E) = \frac{1}{k_B T_g} \exp\left(-\frac{E}{k_B T_g}\right)$$

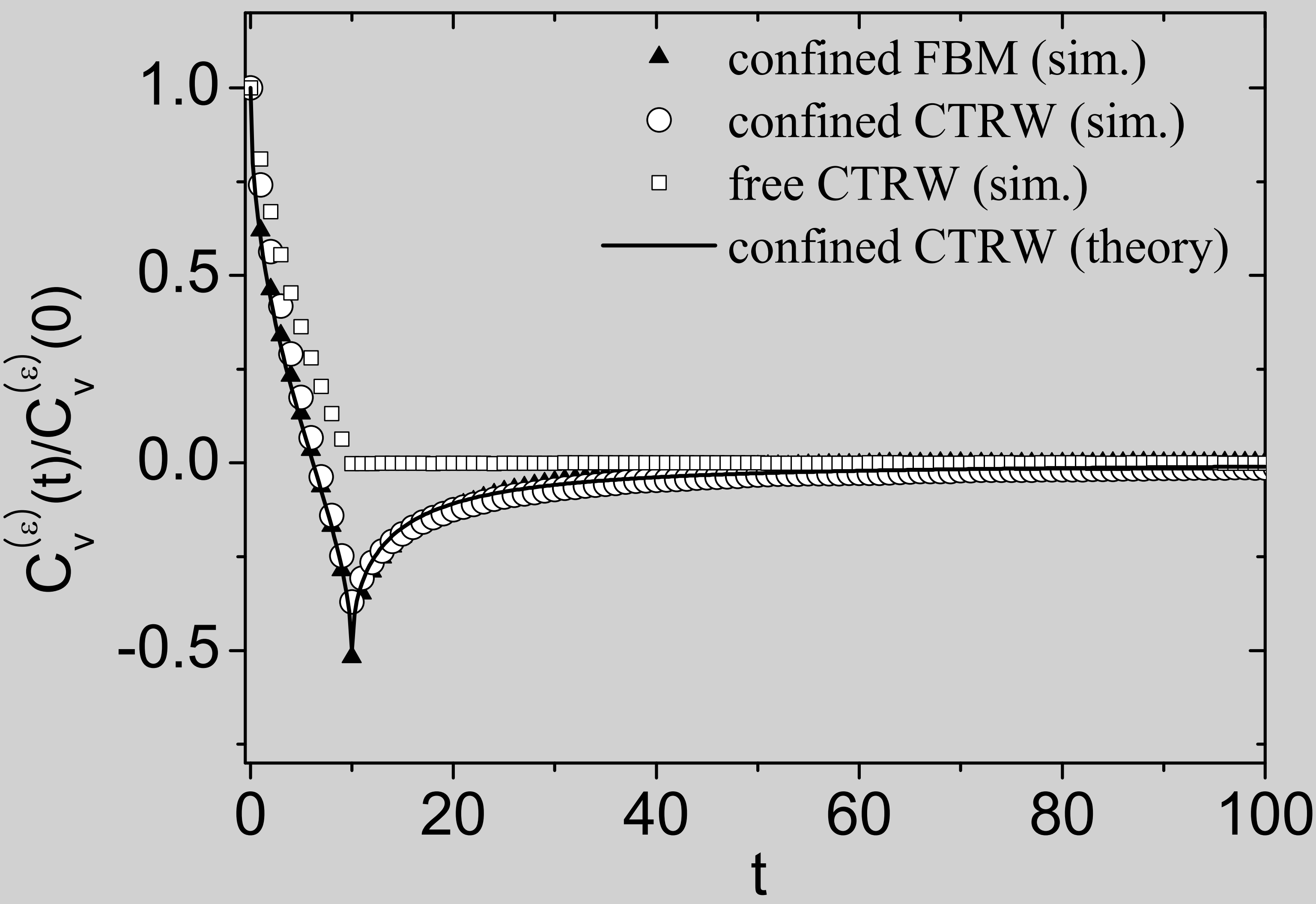
By Arrhenius' law for escape, $\tau_x = \exp(E_x/k_B T)$, we obtain

$$\psi(t) \simeq \frac{T}{T_g} t^{-(1+T/T_g)}$$



JP Bouchaud & DS Dean, J Phys I (1995); C Monthus & JP Bouchaud, J Phys A (1996)

Velocity autocorrelation function: CTRW vs FBM

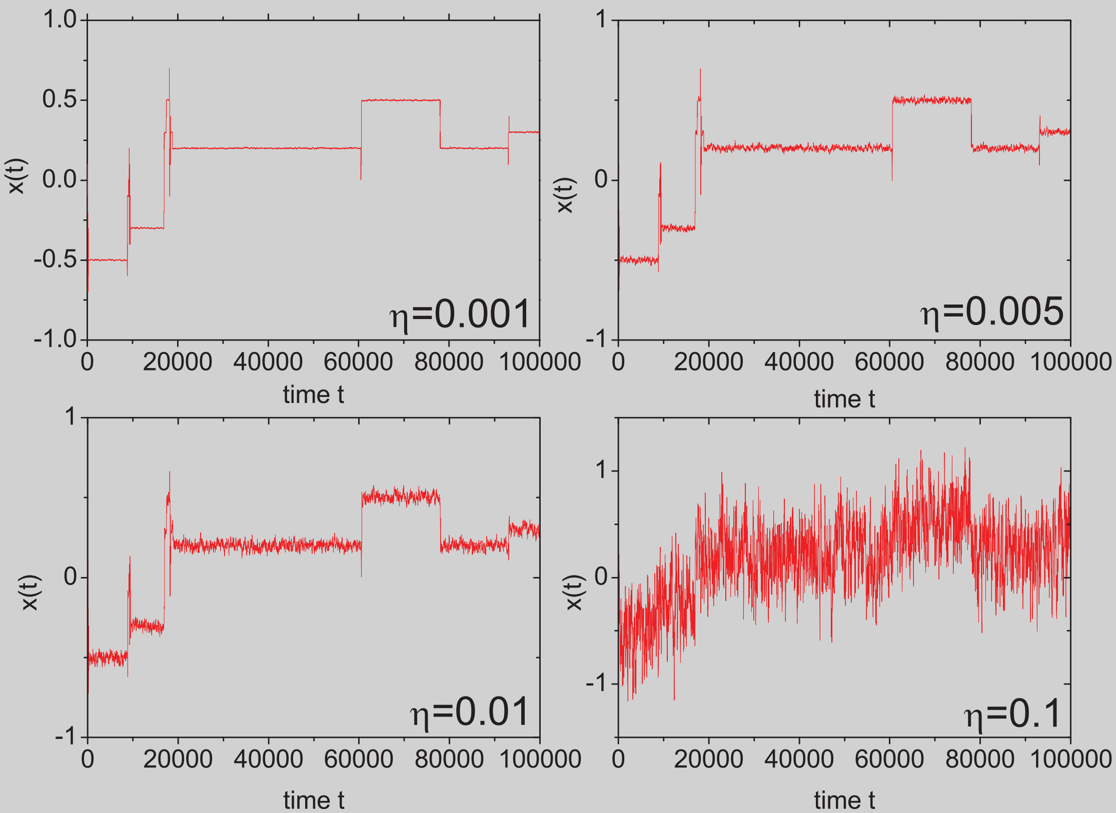


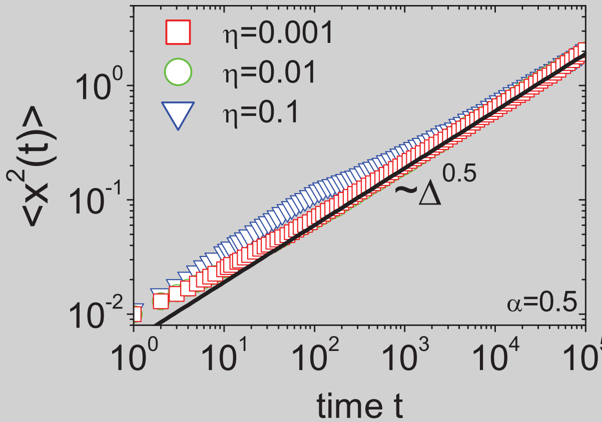
S Burov, J-H Jeon, RM & E Barkai, PCCP (2011)

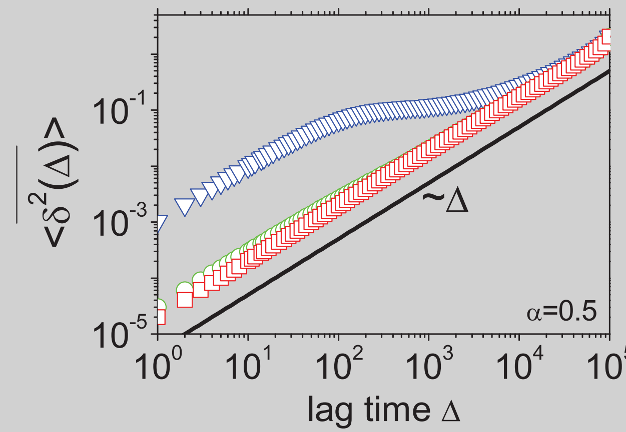
Noisy CTRW processes /w OU-noise

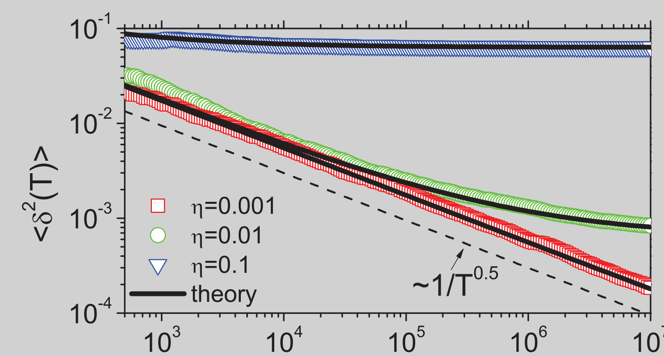
$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)}t^\alpha + \frac{\eta^2 D}{k}\left(1-e^{-2kt}\right)$$

$$\left\langle \overline{\delta^2(\Delta)}\right\rangle \sim \frac{2K_\alpha}{\Gamma(1+\alpha)}\frac{\Delta}{T^{1-\alpha}} + \frac{2\eta^2 D}{k}\left(1-e^{-k\Delta}\right)$$









Correlated Continuous Time Random Walks

Waiting time ψ_i is conditioned by waiting time ψ_{i-1}:

$$P(\psi_i, \psi_{i-1}) = P(\psi_i|\psi_{i-1})P(\psi_{i-1})$$

By iteration this means that ψ_i depends on all previous waiting time:

$$P(\psi_i, \psi_{i-1}, \ldots, \psi_0) = \prod_{n=1}^{i} P(\psi_n|\psi_{n-1})P(\psi_0)$$

Problem: difficult to evaluate

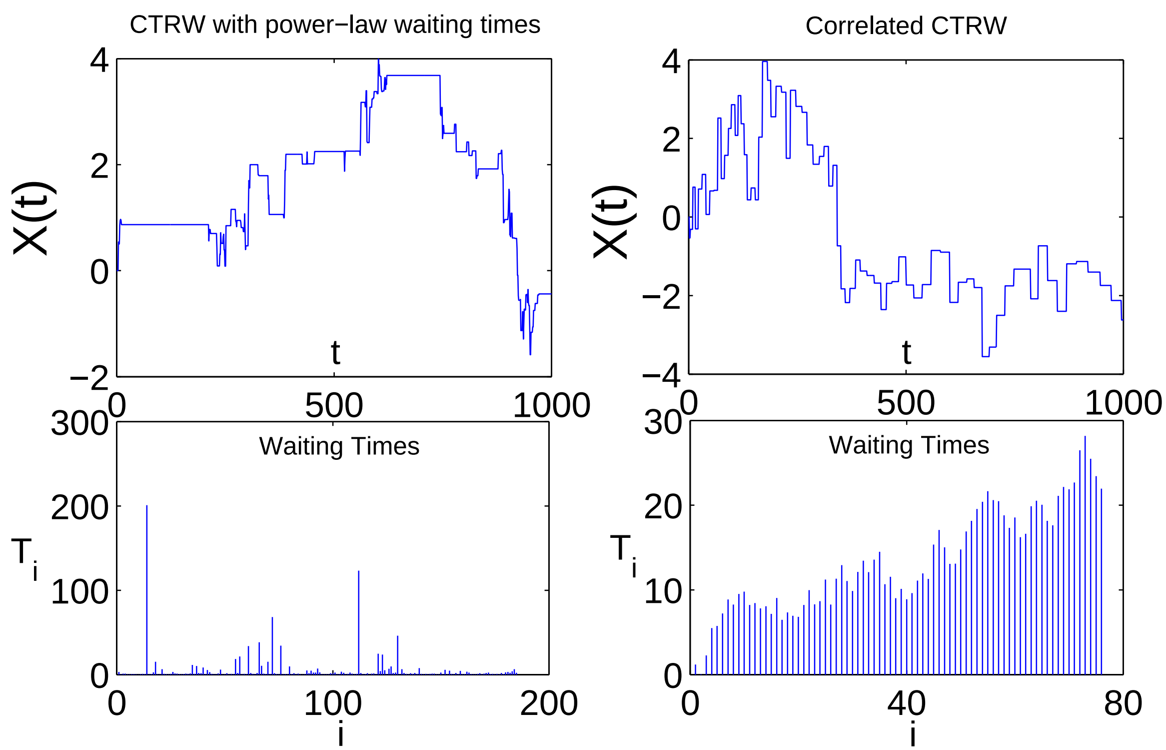
Question: can we find a more direct way of introducing correlations?

Montero & Masoliver, PRE (2007)

Correlated CTRW: coupling of waiting times

Correlated waiting times: $\tau_i = |\tau_{i-1} + \xi_i|$, such that $\langle \tau \rangle \simeq t^{1/\alpha}$

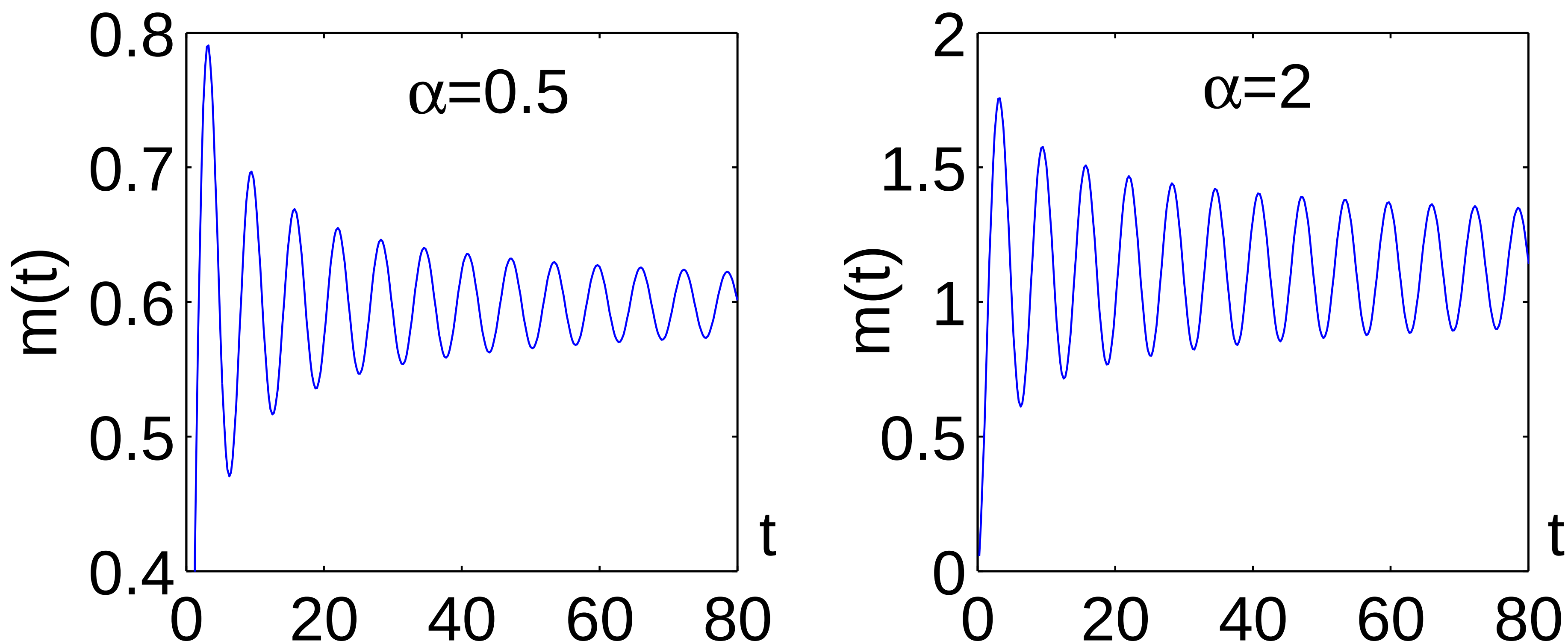
$$\left\langle \overline{\delta^2(\Delta, T)} \right\rangle \simeq \frac{2K_\gamma \Delta}{T^{1-\alpha/(1+\alpha)}}, \qquad \langle x^2(t) \rangle = 2K_\gamma t^\gamma, \quad \gamma = \frac{\alpha}{\alpha+1} \in (0, 2/3]$$



V Tejedor & RM, JPA FTC (2010); M Magdziarz, RM, W Szczotka & P Zebrowski JSTAT (2012), PRE (2012)

CCTRW: Ageing behaviour

Ageing response to sinusoidal force:



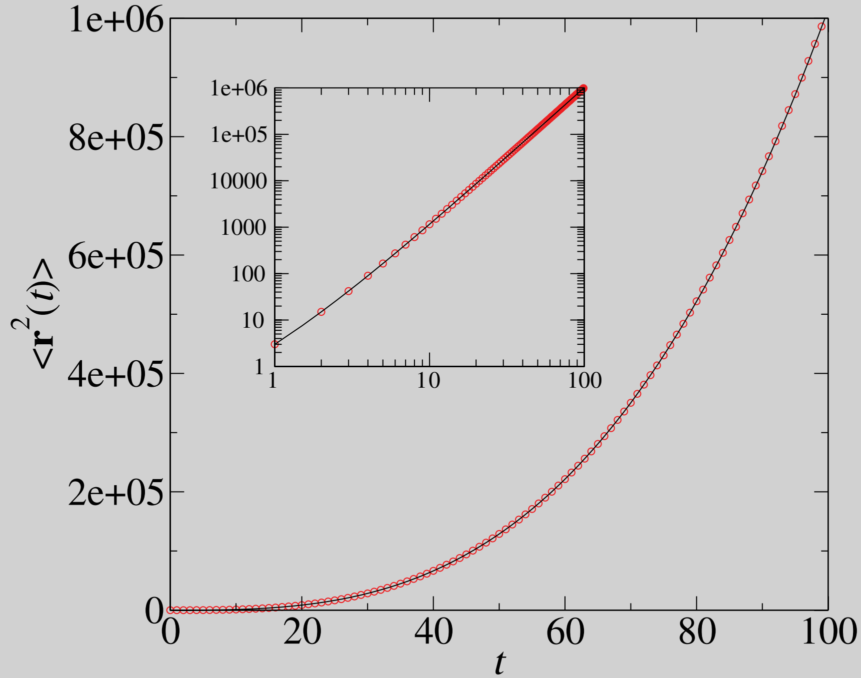
M Magdziarz, RM, W Szczotka & P Zebrowski JSTAT (2012), PRE (2012)

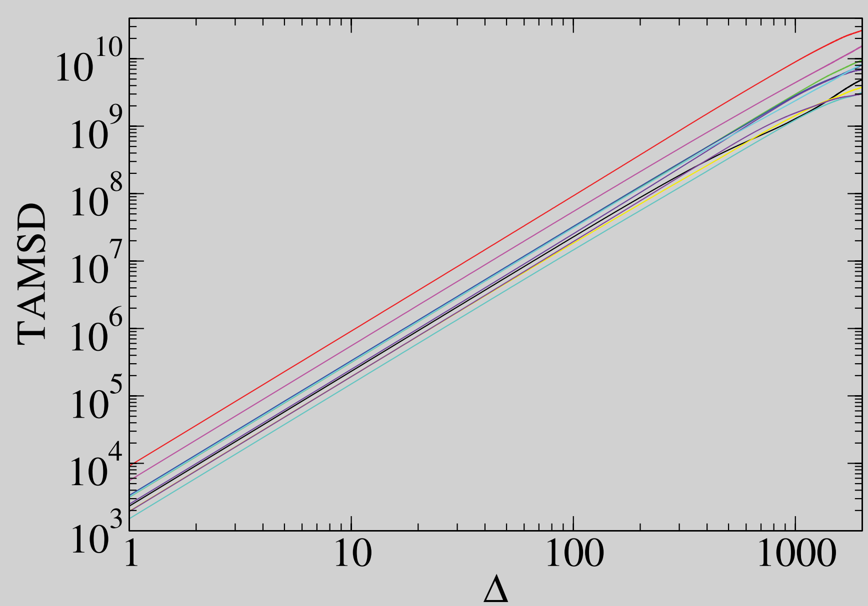
Coupling of jump lengths

Correlated jump lengths: $\lambda_i = \lambda_{i-1} \pm \xi_i$

Gaussian distribution of jump lengths:

$$\langle x^2(t) \rangle = \frac{t(t+1)(2t+1)\sigma^2}{4} \quad \Leftrightarrow \quad \left\langle \overline{\delta^2(\Delta, T)} \right\rangle \sim \frac{3\sigma^2}{4} \Delta^2 T, \quad \Delta \ll T$$

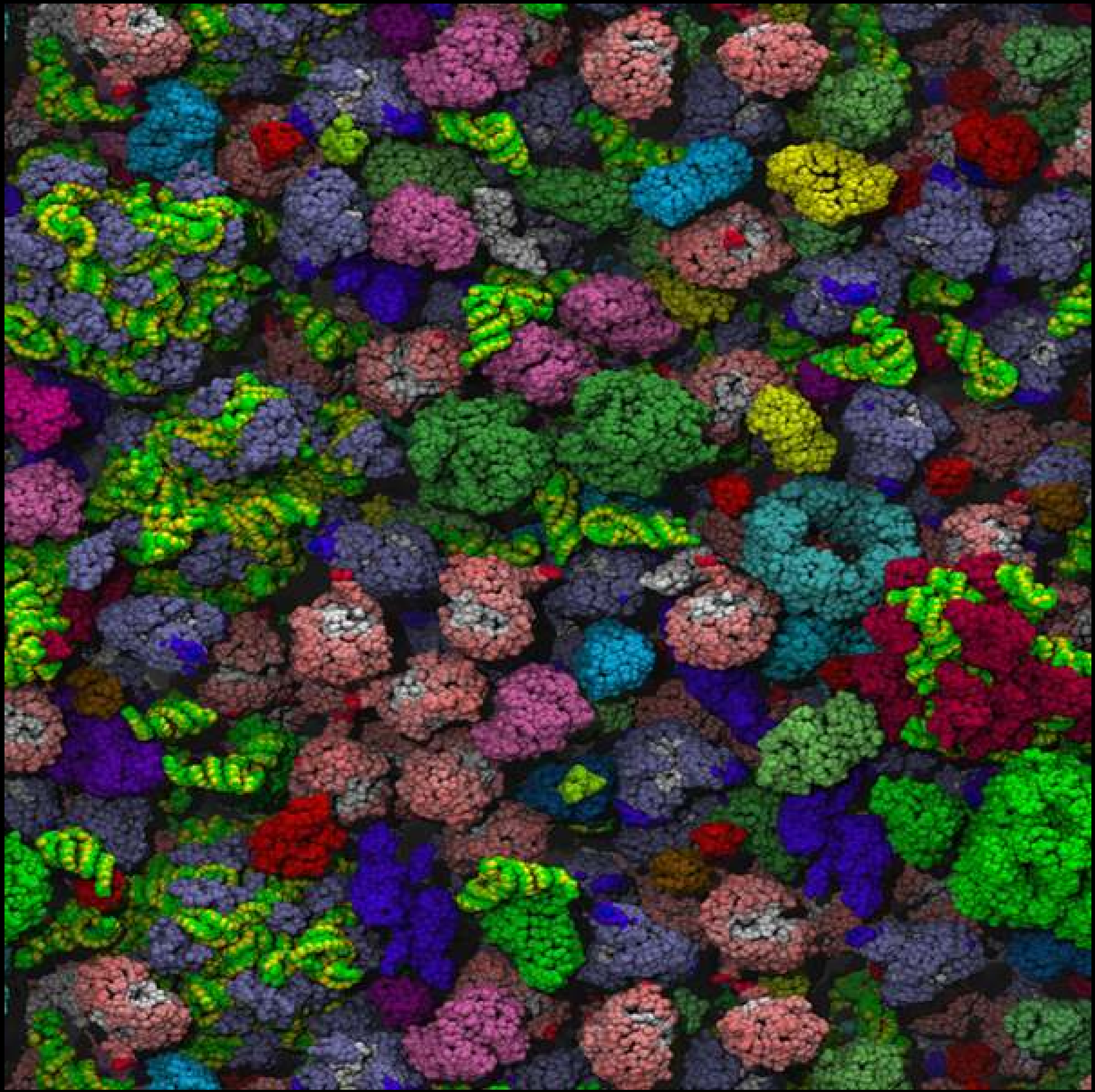




V Tejedor & RM, JPA FTC (2010)


I SÆTTER HJULENE I STÅ
JEG VIL BEHOLDE MIN FRIHED TIL AT KØRE HVOR JEG VIL
HVAD LAVER DEN IDIOT ?
EN KOVENDING!

Macromolecular crowding in living cells



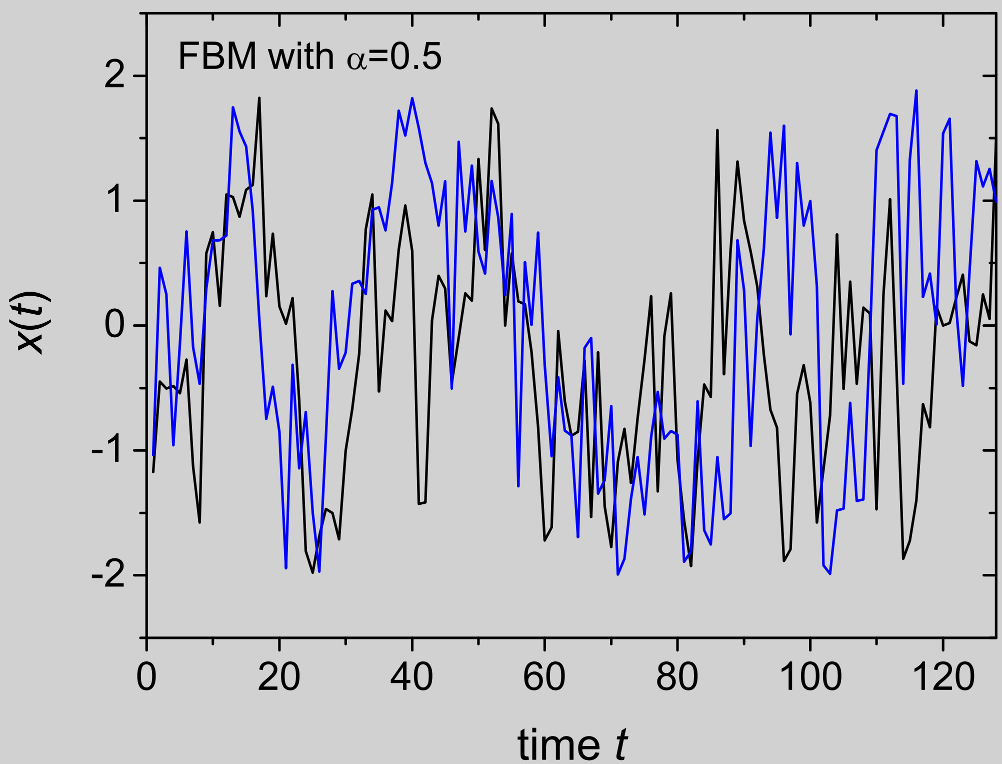
SR McGuffee & AH Elcock, PLoS Comp Biol (2010)

Fractional Brownian motion (viscoelastic environment)

Fractional Gaussian noise: $\langle\xi(t_1)\xi(t_2)\rangle \sim \alpha K_\alpha(\alpha-1)|t_1-t_2|^{\alpha-2}$

Position autocorrelation: $\langle x(t_1)x(t_2)\rangle = K_\alpha\Big(t_1^\alpha + t_2^\alpha - |t_1-t_2|^\alpha\Big);\;\; t_1, t_2 > 0$

Ergodicity is satisfied: $\left\langle\overline{\delta^2(\Delta,T)}\right\rangle \sim 2K_\alpha\Delta^\alpha = \left\langle x^2(\Delta)\right\rangle$



Mandelbrot & van Ness, SIAM (1968); W Deng and E Barkai, PRE (2009); RM et al, Acta Phys Polon B (2009)

Viscoelastic diffusion $\langle \mathrm{r}^2(t)\rangle \simeq K_\alpha t^\alpha$ is asymptotically ergodic

Fractional Langevin equation:

$$\ddot{\mathbf{r}}(t) + \int_0^t \eta(t-t')\dot{\mathbf{r}}(t')dt' = \boldsymbol{\zeta}(t)$$

with $\eta(t) = \sum_{i=1}^N a_i(k)e^{-\nu_i t} \to t^{-\alpha}$ &
$\langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{ij}k_B T\eta(|t-t'|)$, Gauss

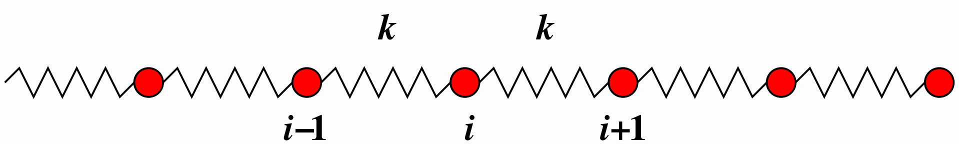
Fractional Brownian motion:

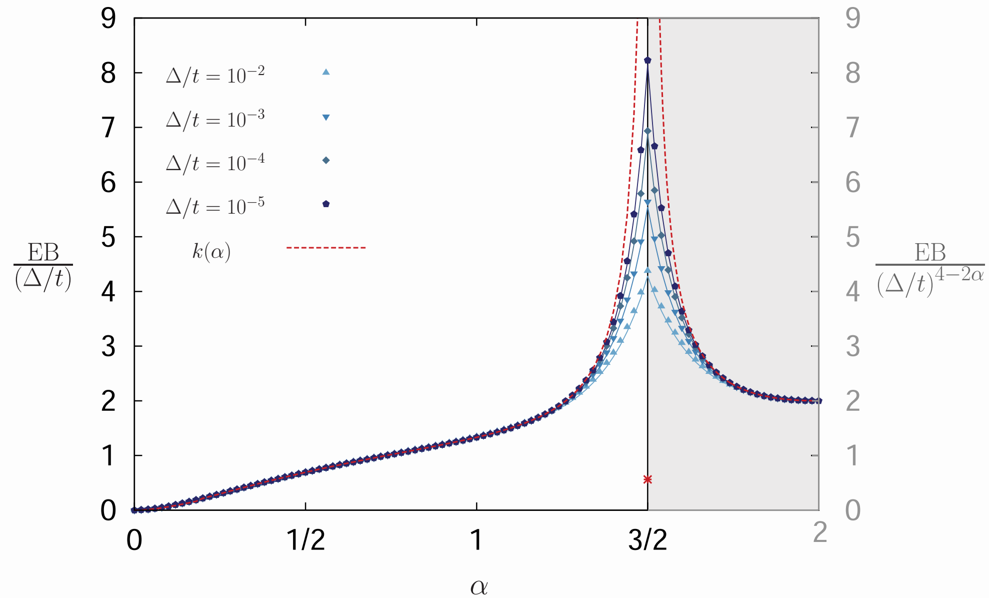
$$\dot{\mathbf{r}}(t) = \boldsymbol{\zeta}(t)$$

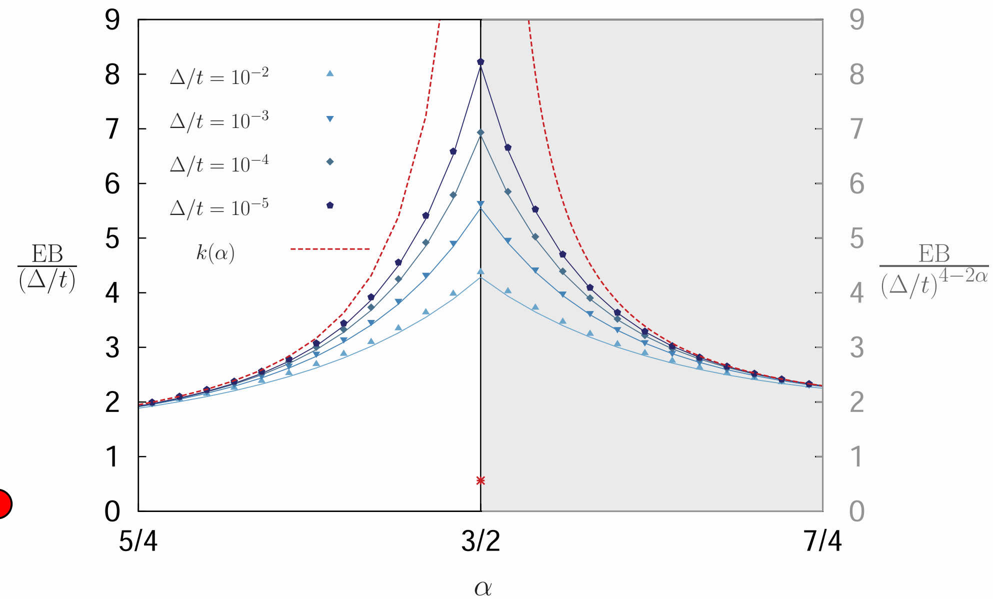
In both cases: $\lim_{T\to\infty}\overline{\delta^2(\Delta)} = \langle \mathrm{r}^2(\Delta)\rangle$

Ergodicity breaking parameter:

$$\mathrm{EB} = \langle \xi^2\rangle - 1, \xi = \overline{\delta^2(\Delta)}\Big/\left\langle\overline{\delta^2(\Delta)}\right\rangle$$

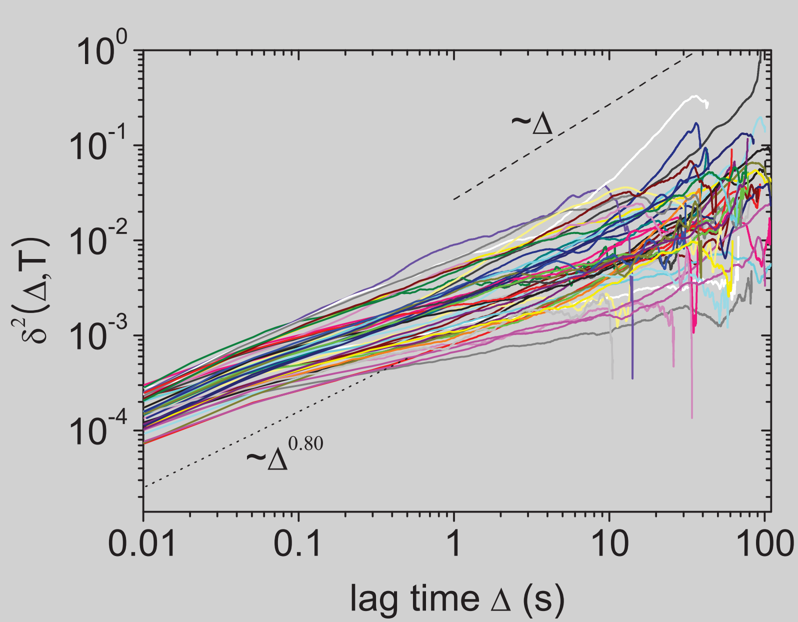


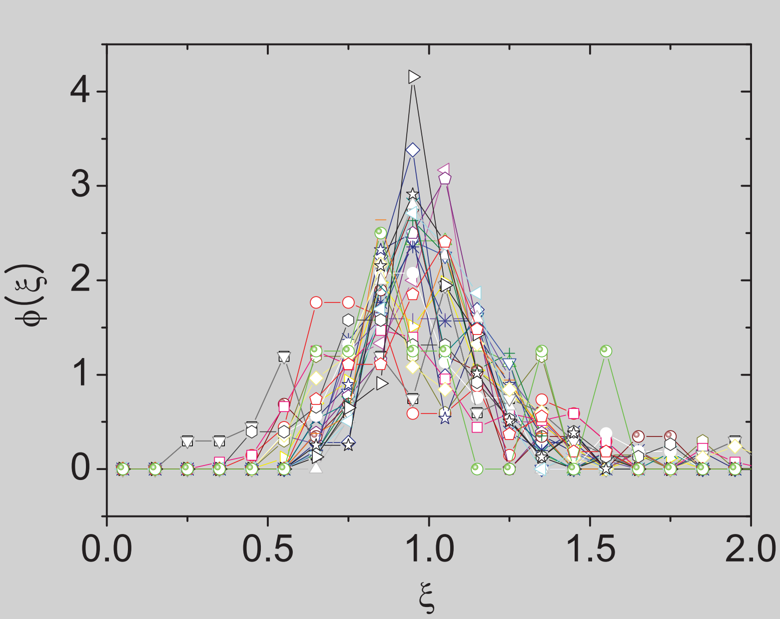




W Deng & E Barkai, PRE (2009); JH Jeon & RM, PRE (2010); M Schwarzl, A Godec & RM, Sci Rep (2017)

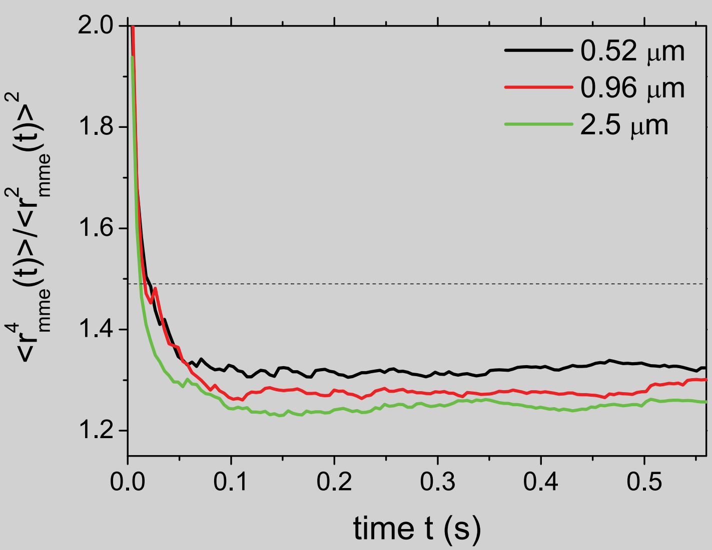
Passive motion of submicron tracers in cells is viscoelastic

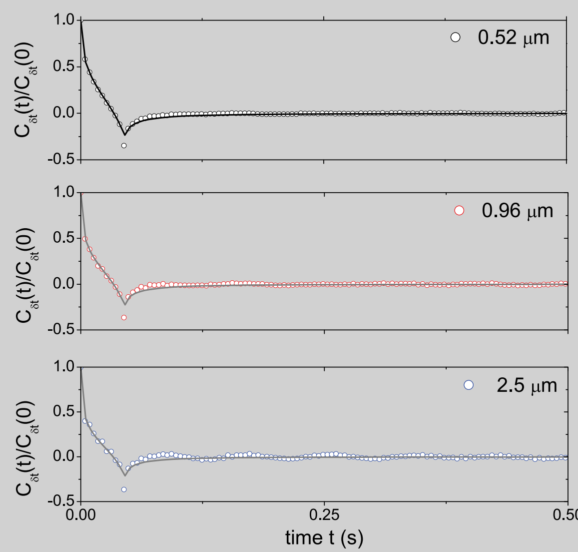


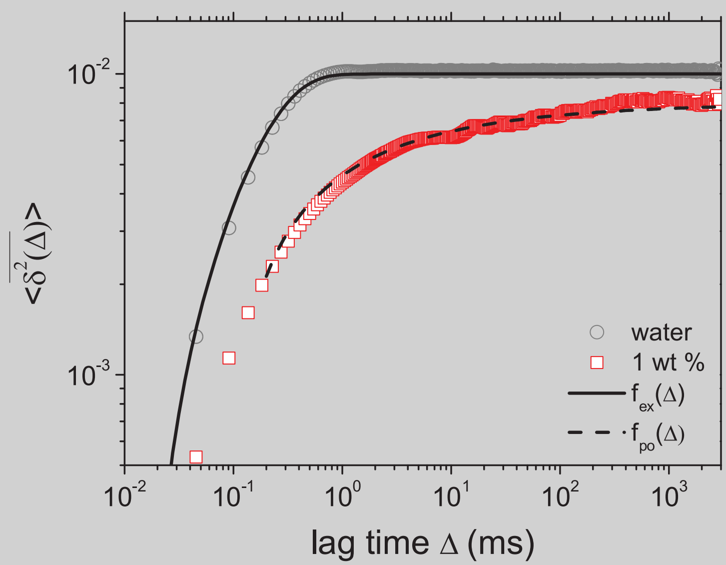


Lipid granules in living yeast cells ←

Tracer beads in wormlike micellar solution →

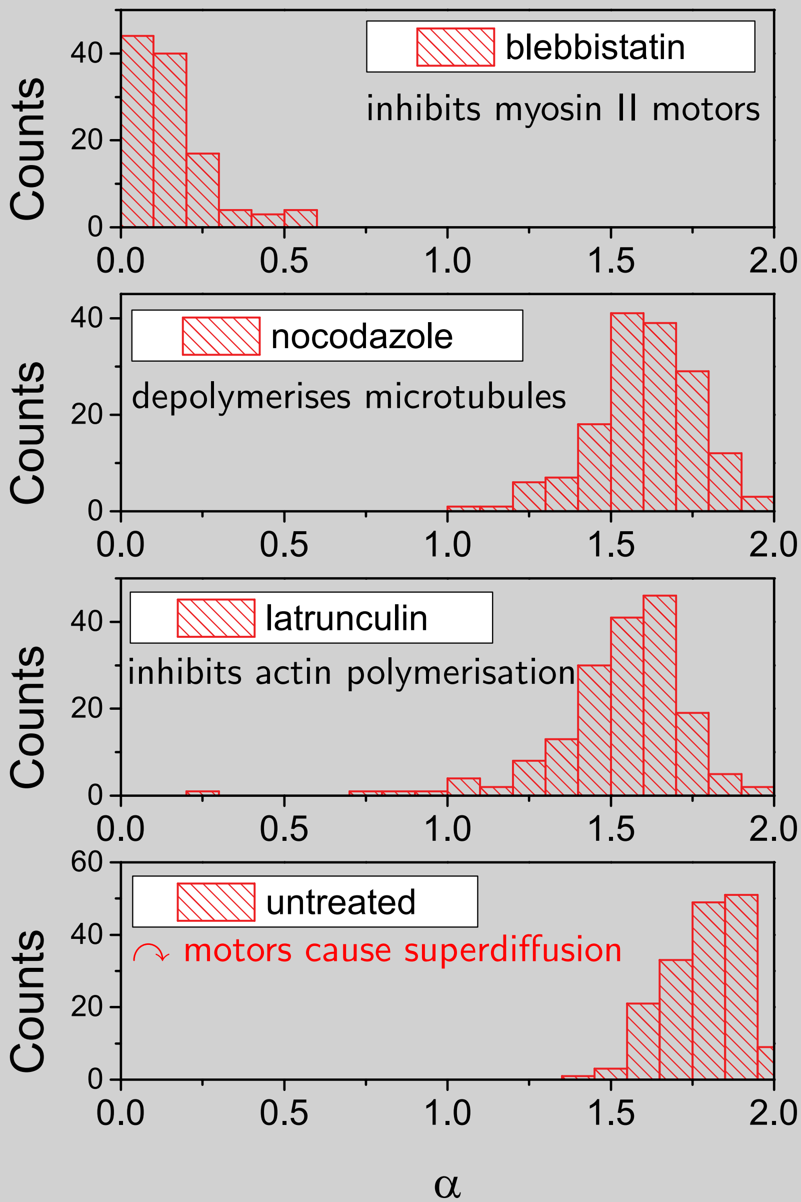


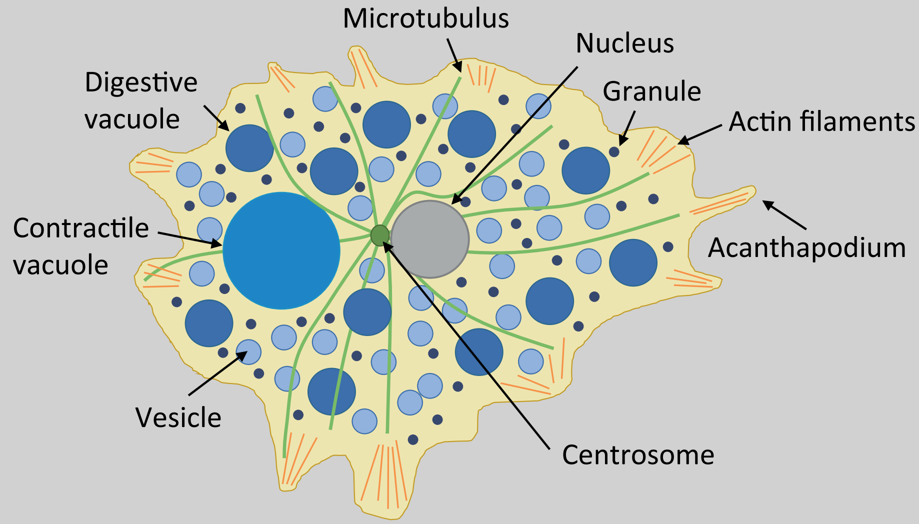


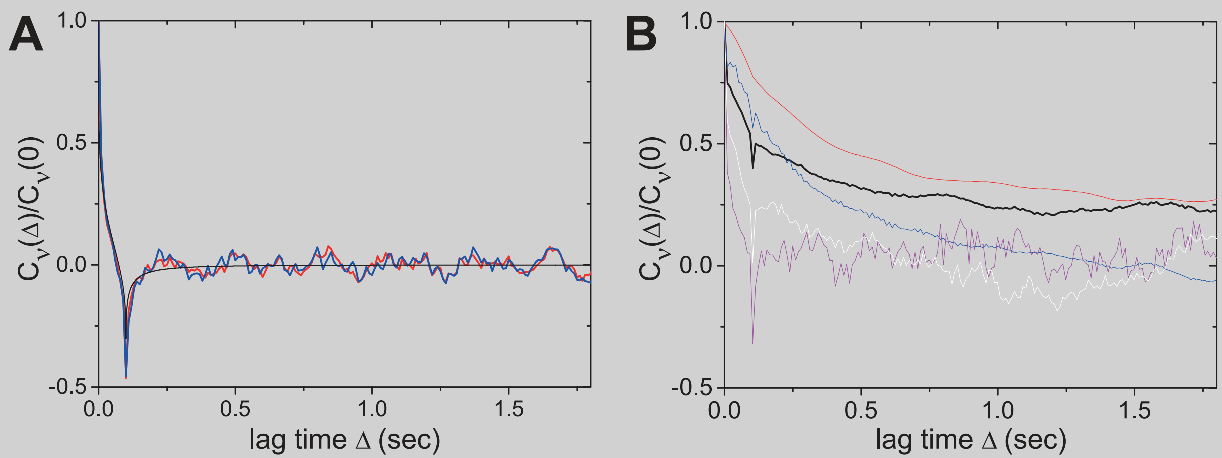


JH Jeon, . . . L Oddershede & RM, PRL (2011); JH Jeon, N Leijnse, L Oddershede & RM, NJP (2013); JH Jeon & RM, PRE (2012)

Superdiffusion in supercrowded Acanthamoeba castellani



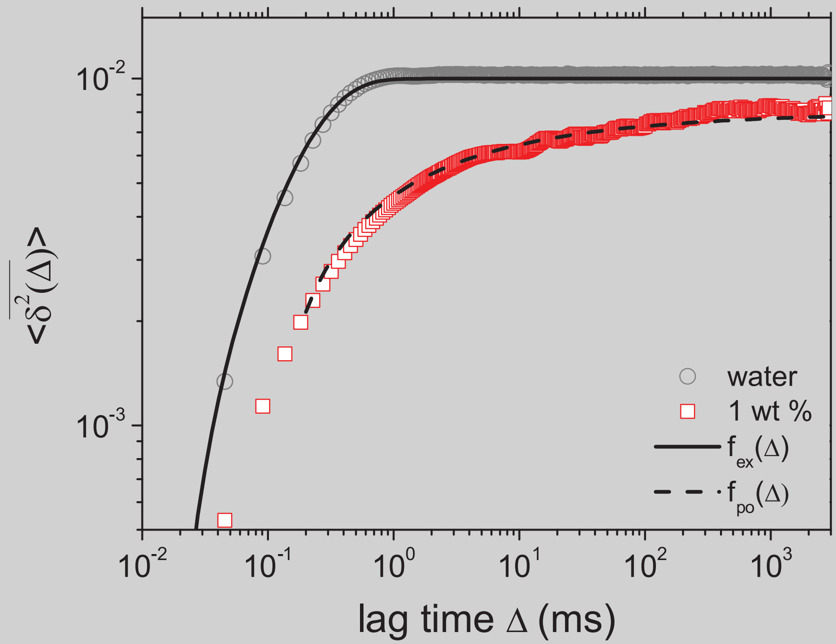




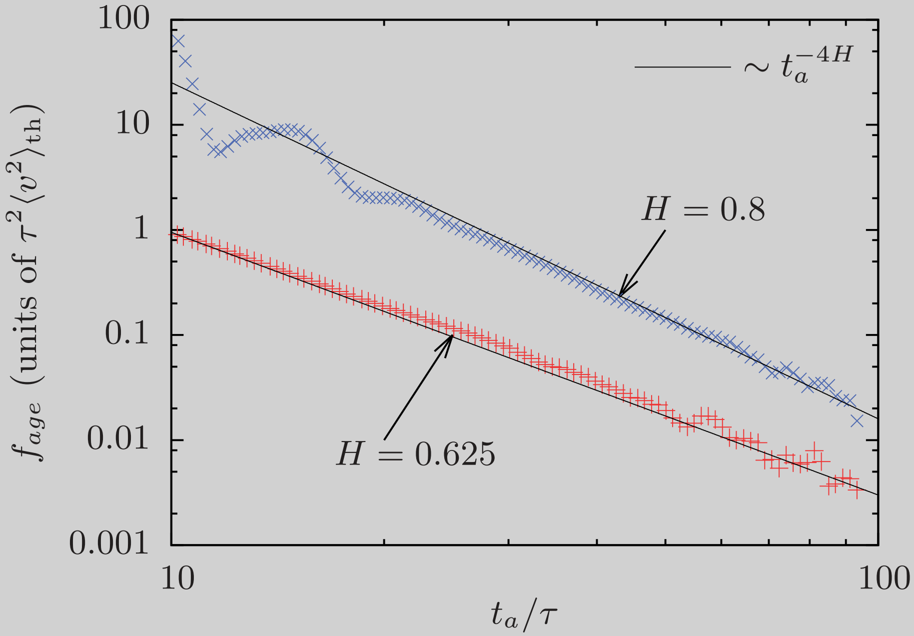
JF Reverey, J-H Jeon, H Bao, M Leippe, RM & C Selhuber-Unkel, Sci Rep (2015)

Viscoelastic anomalous diffusion: transient WEB & ageing

Unbounded motion is ergodic: $\overline{\delta^2(\Delta)} = \langle \mathbf{r}^2(\Delta)\rangle$



The motion does not age



Transient WEB in confiment:

$$\overline{\delta^2(\Delta)} \sim 2\langle x^2\rangle_{\rm th}\left(1-\frac{\gamma}{k\Delta^{2-\alpha}}\right)$$

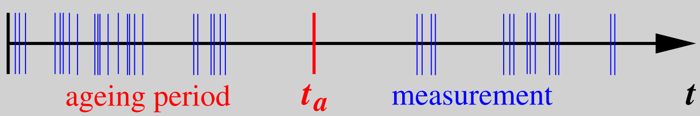
[J-H Jeon, N Leijnse, L Oddershede & RM, New J Phys (2013),
J-H Jeon & RM, PRE (2012)]

W Deng & E Barkai, PRE (2009)

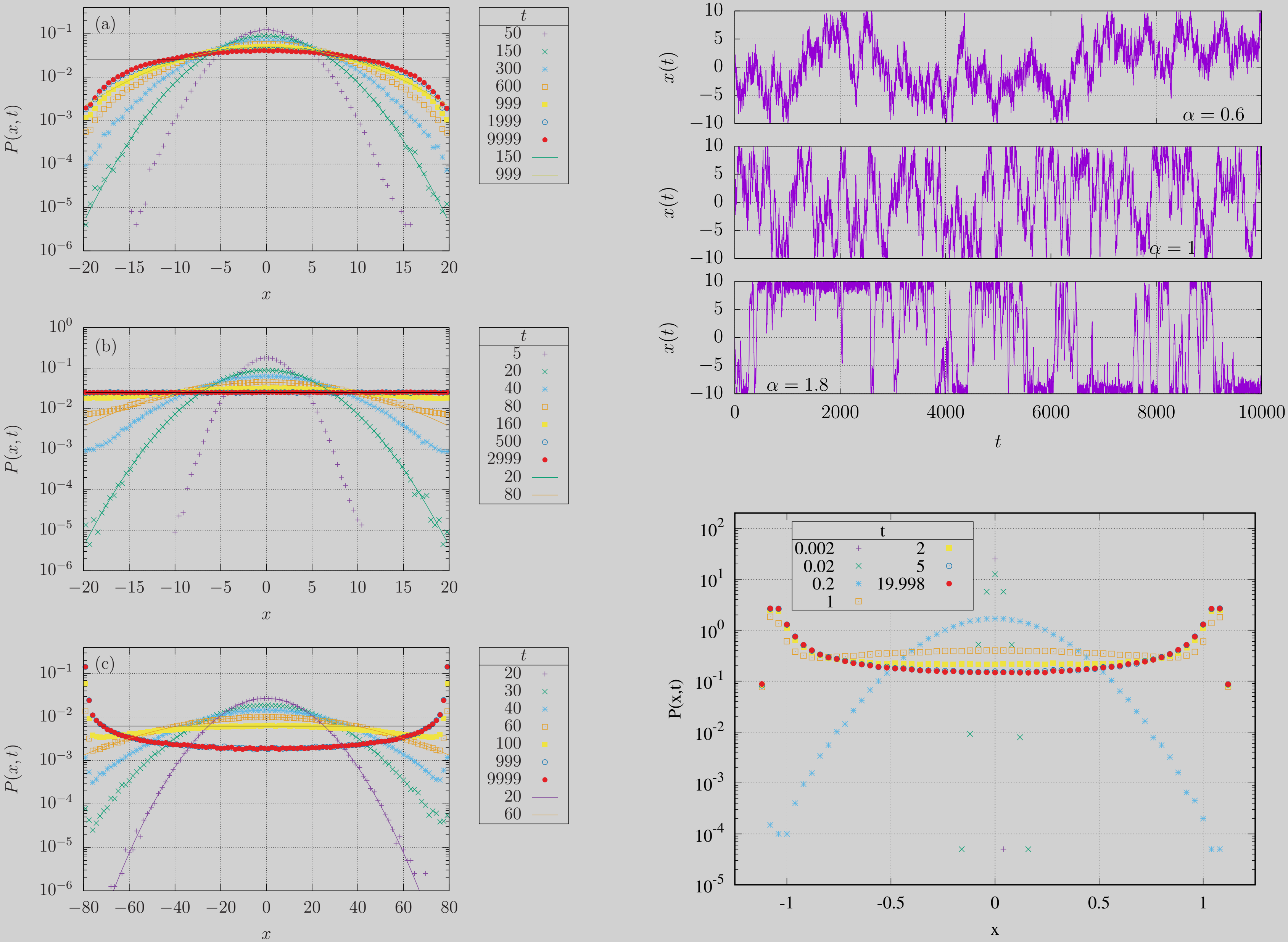
Transient ageing:

$$\overline{\delta^2(\Delta)} = f_{\rm st}(\Delta) + f_{\rm age}(\Delta, t_a, T)$$

[J Kursawe, J Schulz & RM, PRE (2013)]

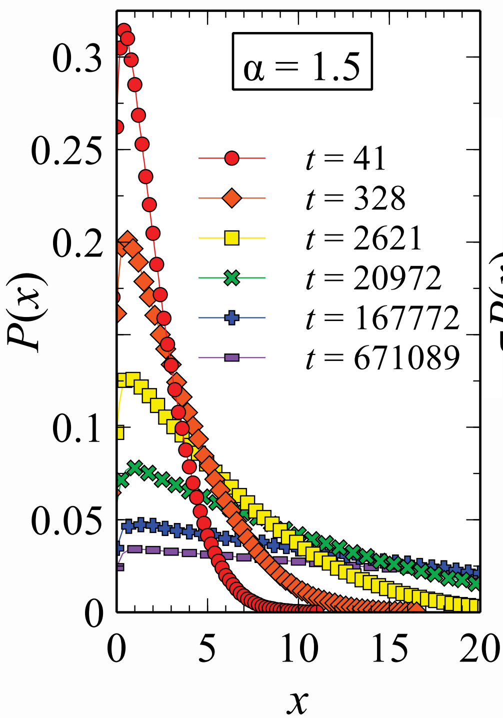


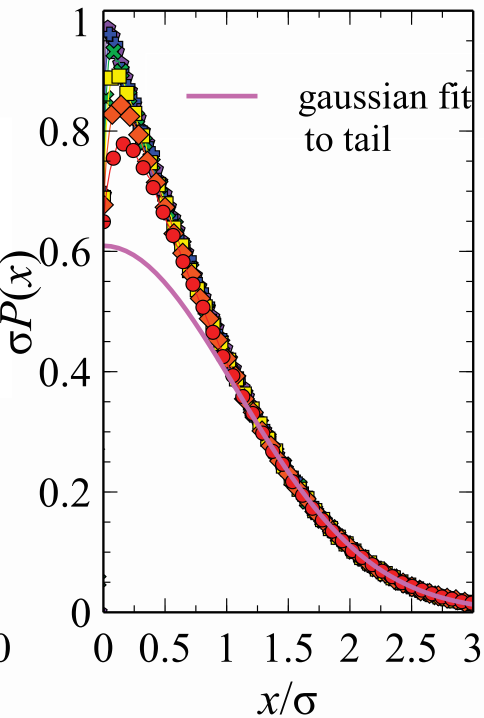
FBM: accretion & depletion effects near boundaries

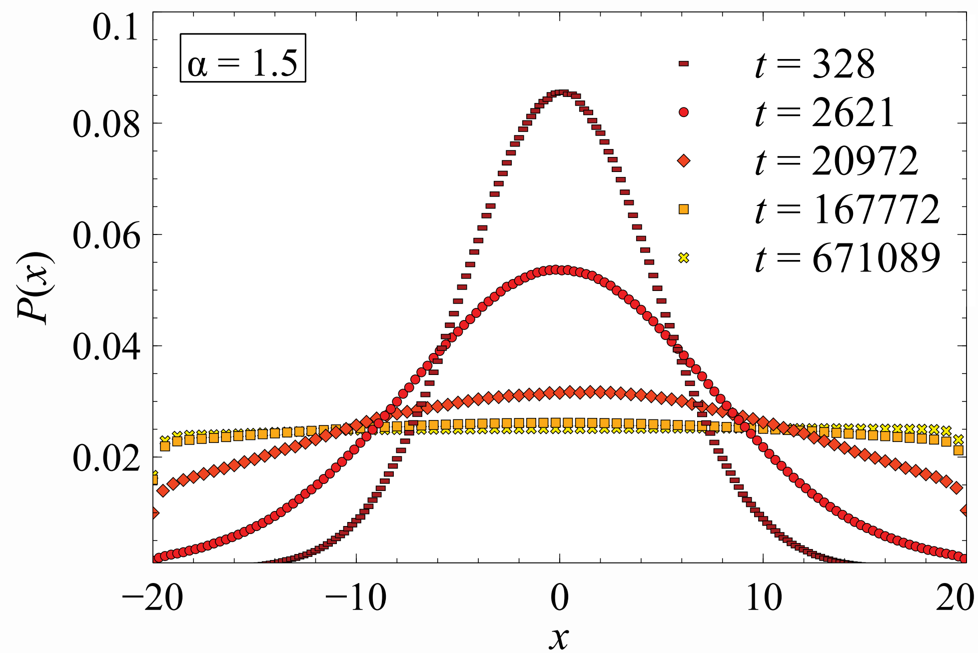


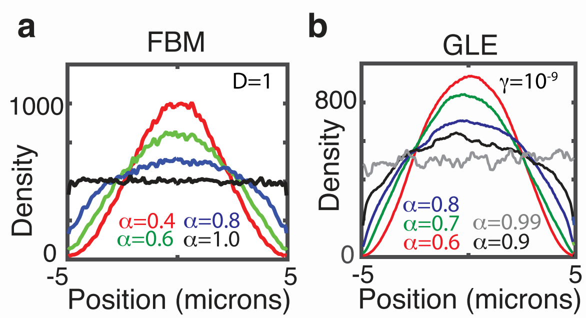
T Guggenberger, G Pagnini, T Vojta & RM, NJP (2019); see also AHO Wada & T Vojta, PRE (2018)

FLE shows equilibrated behaviour









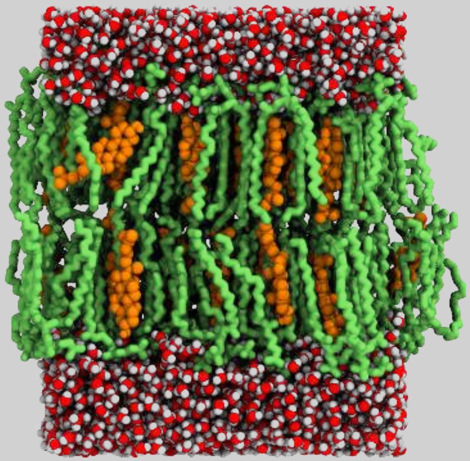
Holmes, BPJ (2019) using hard BC

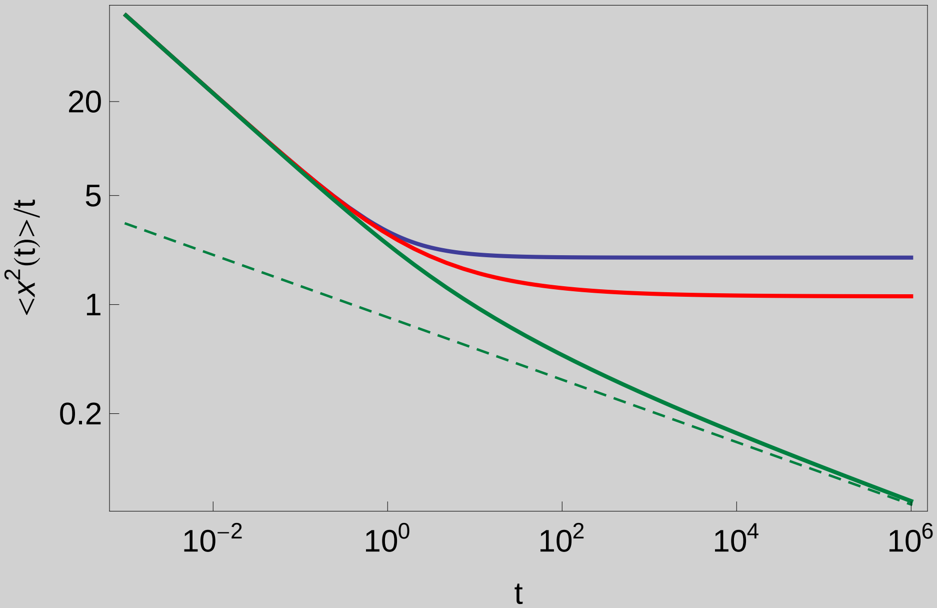
T Vojta, S Skinner & RM (2019)

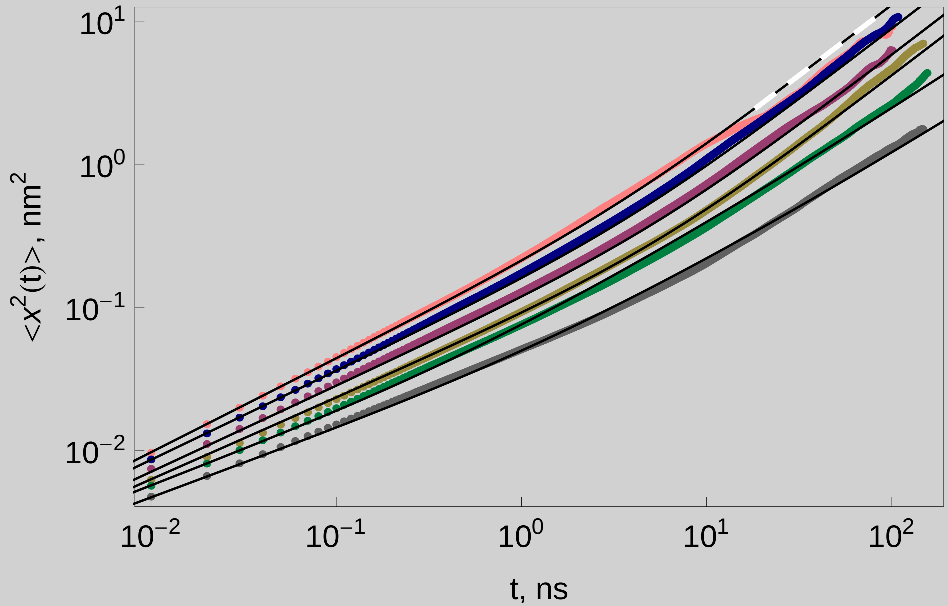
Tempered FLE motion: crossover to faster diffusion

Tempered fractional Gaussian noise:

$$\langle \xi(t)\xi(t+\tau)\rangle = \begin{cases} \dfrac{C}{\Gamma(2H-1)}\tau^{2H-2}e^{-\tau/\tau_\star} \\ \dfrac{C}{\Gamma(2H-1)}\tau^{2H-2}\left(1+\dfrac{\tau}{\tau_\star}\right)^{-\mu} \end{cases}$$

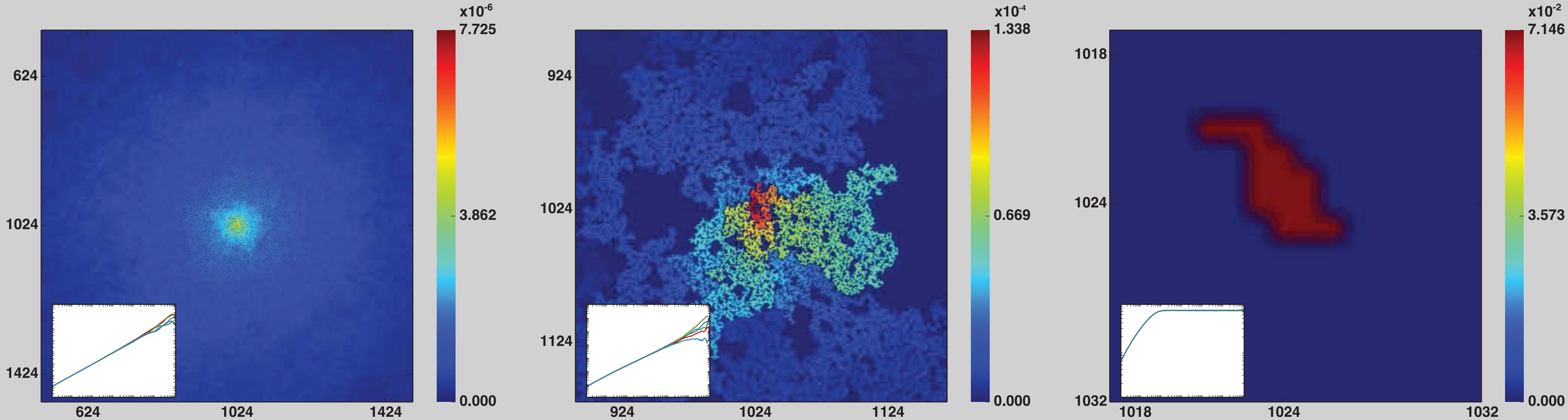






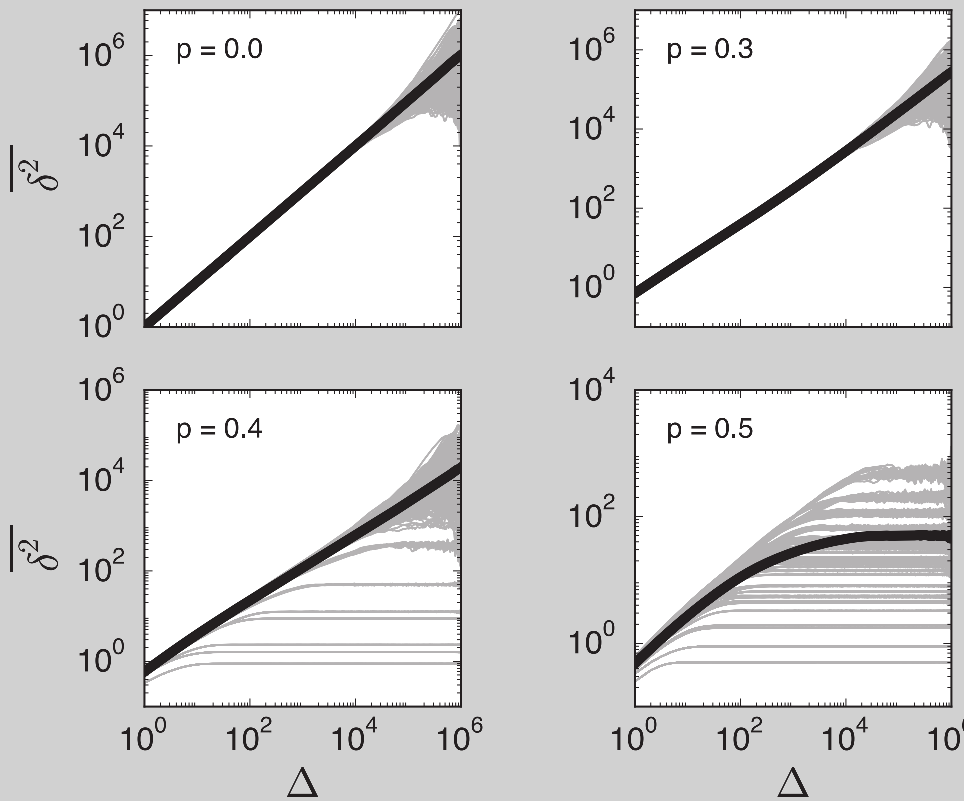
D Molina-Garcia, T Sandev, H Safdari, G Pagnini, AV Chechkin & RM, NJP (2018)

Geometry induced strong non-ergodicity in random fractals



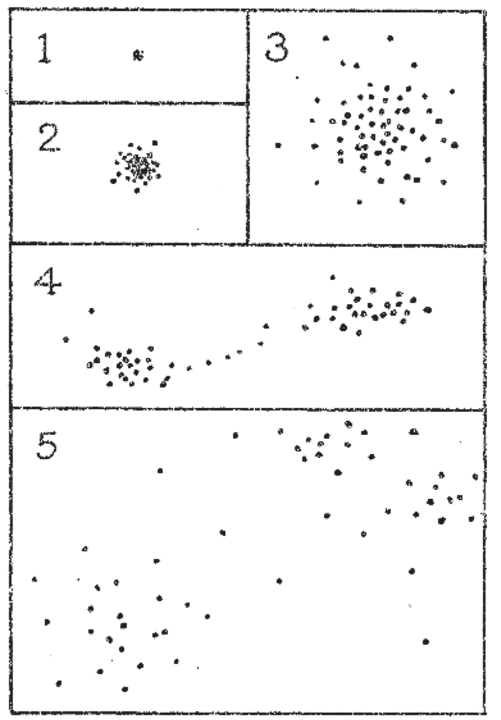
$$\left\langle \widetilde{\overline{\delta^2(\Delta)}} \right\rangle = \sum_{i,\zeta} \frac{\overline{\delta^2_{i,\zeta}(\Delta)}}{N \times N_\zeta} = \langle \widetilde{\mathbf{r}^2(t)} \rangle$$

Non-ergodic: $\lim_{T\to\infty} \overline{\delta^2(\Delta)}$



Y Mardoukhi, J-H Jeon & RM, PCCP (2015)

Richardson (relative) diffusion in turbulence: $\langle l^2(t)\rangle \simeq t^3$



FIGS. 1–5.

Brown

Turbulent spreading

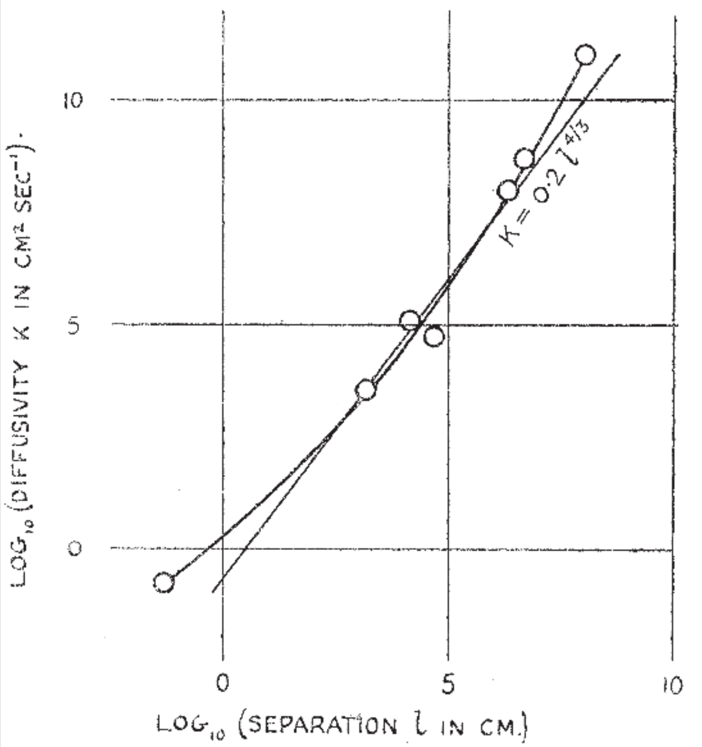


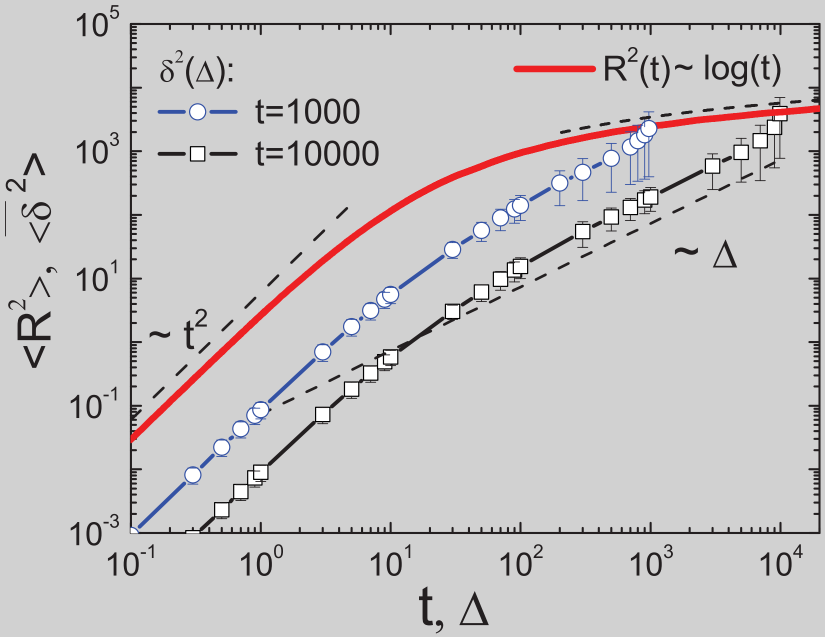
FIG. 8.

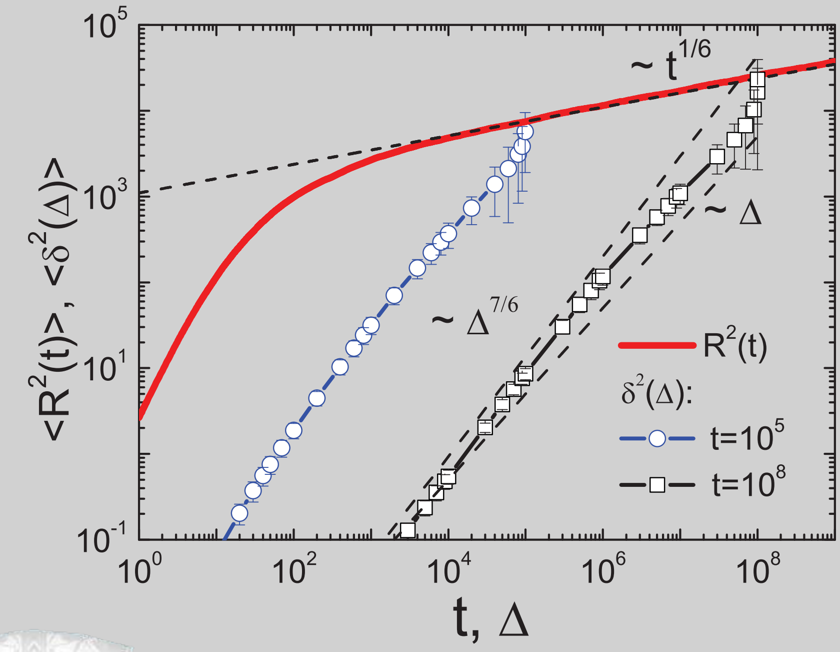
$$\frac{\partial q}{\partial t} = \varepsilon \frac{\partial}{\partial l}\left(l^{4/3} \frac{\partial q}{\partial l} \right), \quad \varepsilon \sim 0.4\,\mathrm{cm}^{2/3}\mathrm{sec}^{-1}$$ Richardson

$$\frac{\partial q}{\partial t} = \bar{\varepsilon} t^2 \frac{\partial^2}{\partial l^2} q(l,t)$$ Batchelor

LF Richardson, Proc Roy Soc London Ser A (1926); GK Batchelor, Proc Cambridge Philos Soc (1952)

WEB in granular gas & SBM as mean field theory

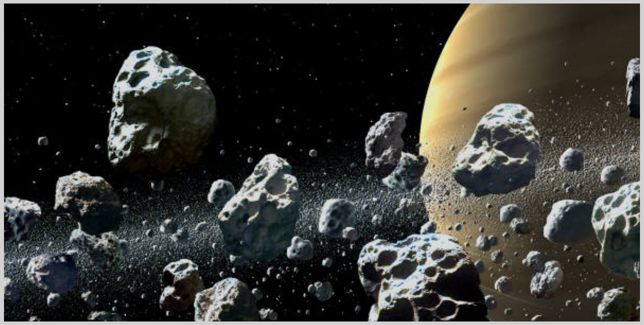




Haff's law: $\mathscr{T}(t) = \mathscr{T}_0/(1+t/\tau_0)^2$

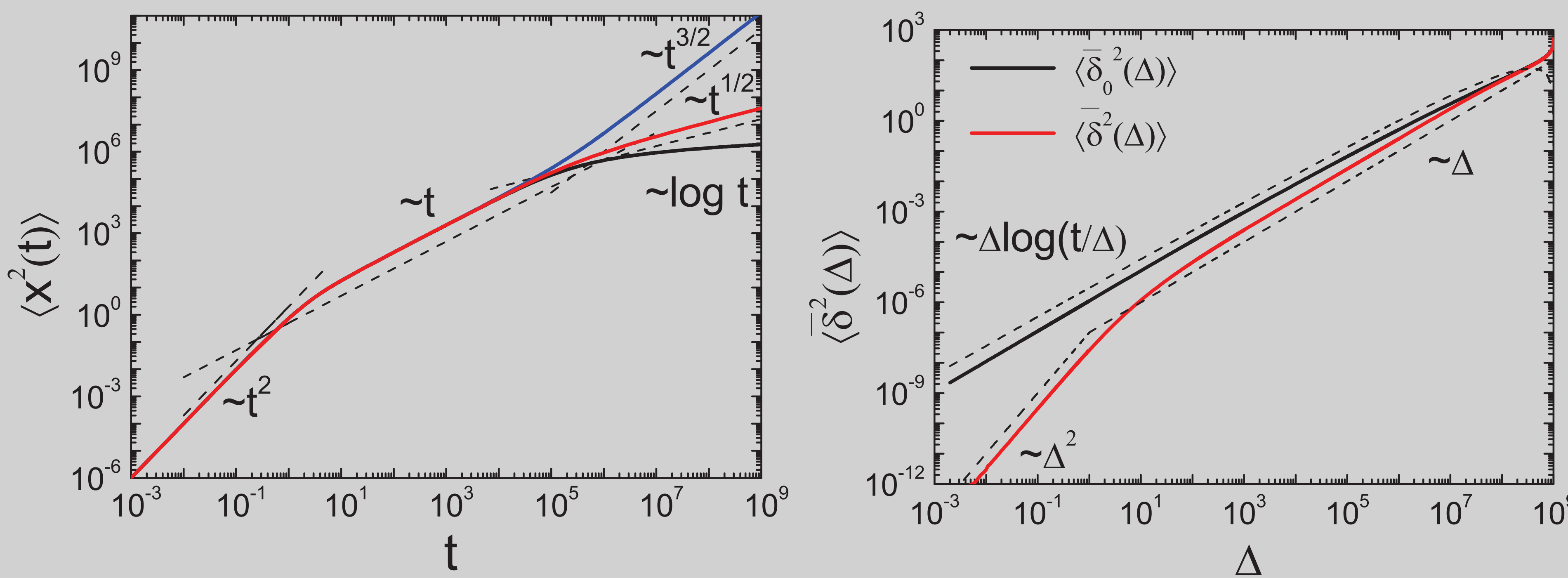
$$\langle \mathbf{r}^2(t)\rangle \sim 6D_0\tau_0\log(1+t/\tau_0)$$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim 6D_0\tau_0\Delta/T$$



A Bodrova, AV Chechkin, AG Cherstvy & RM, NJP (2015); PCCP (2015)

Non-existence of the overdamped limit in slow SBM



Crossover from ballistic to overdamped motion no longer defined by time scale of inverse friction. For small α & ultraslow the SBM overdamped limit is never fulfilled

Ageing case: [H Safdari, A Bodrova, AV Chechkin, AG Cherstvy & RM, PRE (2017)]

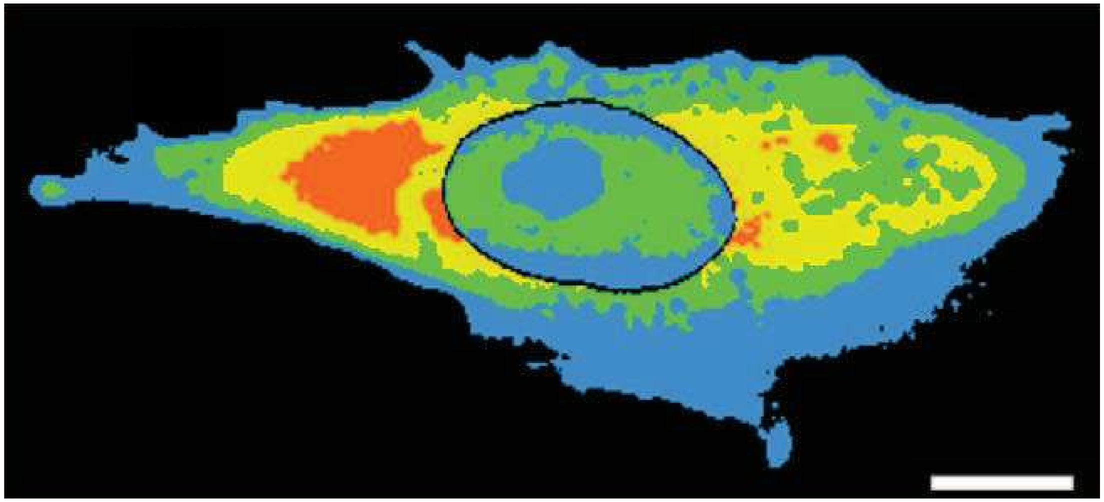
Position dependent diffusivity: heterogeneous diffusion

Markovian diffusion process with $K = K(x)$ à la Richardson:

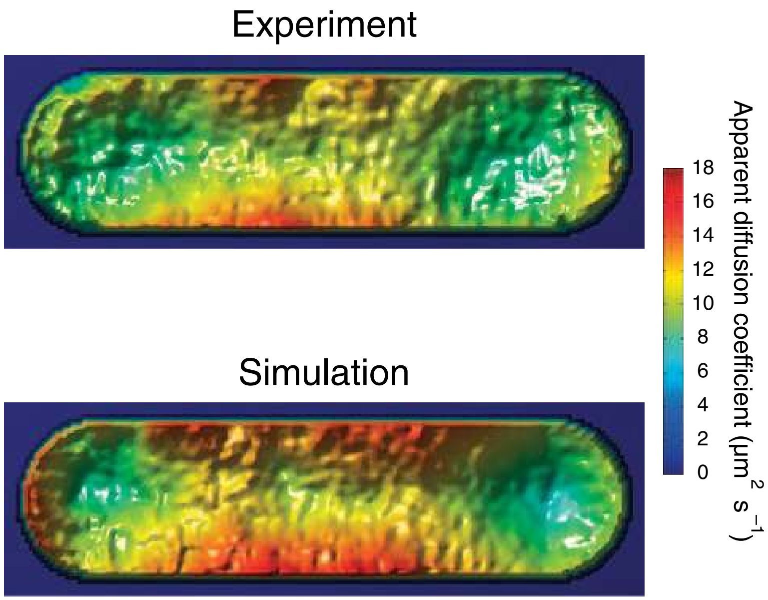
$$K(x) = K_0|x|^\alpha \quad \Longrightarrow \quad \langle x^2(t)\rangle \simeq t^p$$

Multiplicative Langevin equation:

$$\frac{dx(t)}{dt} = \sqrt{D(x)}\xi(t)$$



Kuehn et al, PLoS ONE (2011)



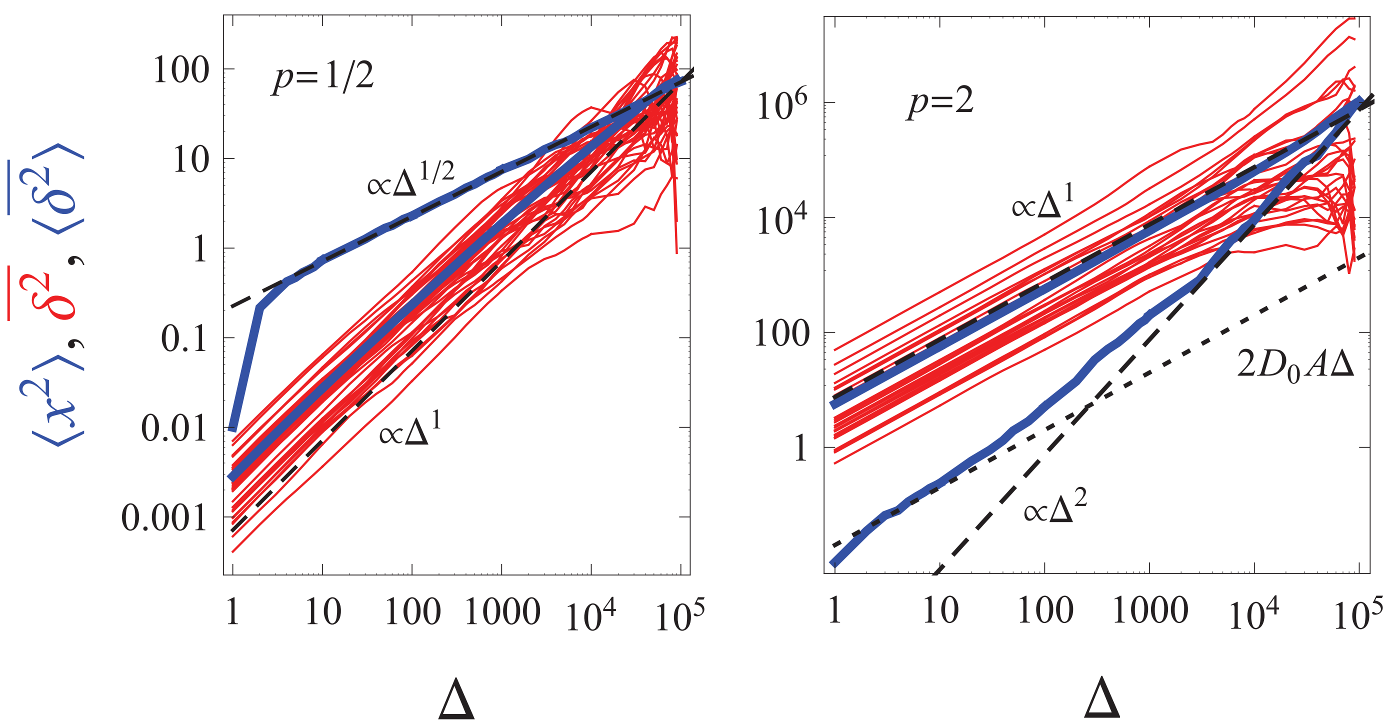
BP English et al, PNAS (2011)

AG Cherstvy, AV Chechkin & RM, New J Phys (2013); PCCP (2013); Soft Matter (2014); J Phys A (2014)

Heterogeneous diffusion processes with $K(x)$

Markovian diffusion process with $K = K(x)$ à la Richardson:

$$K(x) = K_0|x|^\beta \quad \Longrightarrow \quad \langle x^2(t)\rangle \simeq t^p \qquad \left\langle\overline{\delta^2(\Delta)}\right\rangle \simeq \frac{\Delta}{T^{1-p}}, \qquad p = \frac{2}{2-\beta}$$



Many common features with CTRW: PRE (2014), JPA (2014)

AG Cherstvy, AV Chechkin & RM, New J Phys (2013); PCCP (2013); Soft Matter (2014)

Ageing behaviour in anomalous diffusion processes

Continuous time random walk with $\psi(t) \simeq \tau^\alpha / t^{1+\alpha}$:

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle_a = \Lambda_\alpha(t_a/T) \left\langle \overline{\delta^2(\Delta)} \right\rangle \qquad \not\Leftrightarrow \qquad \langle x^2(t) \rangle \simeq \begin{cases} t^\alpha, & t_a \ll t \\ t_a^{\alpha-1} t, & t_a \gg t \end{cases}$$

$\therefore \Lambda_\alpha(z) = (1+z)^\alpha - z^\alpha$ [JHP Schulz, E Barkai & RM, PRL (2013); PRX (2014)]

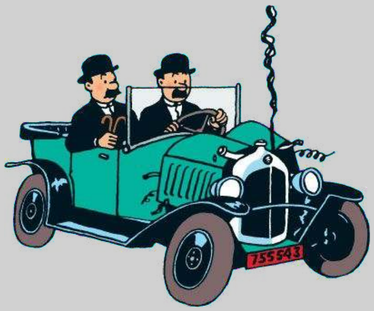
Same functional forms for Scaled Brownian Motion with $K(t) = K_\alpha t^{\alpha-1}$
[H Safdari, AV Chechkin, GR Jafari & RM, PRE (2015)]

and for heterogeneous diffusion processes with $K(x) = K_0 |x|^\alpha$
[AG Cherstvy, AV Chechkin & RM, JPA (2014)]

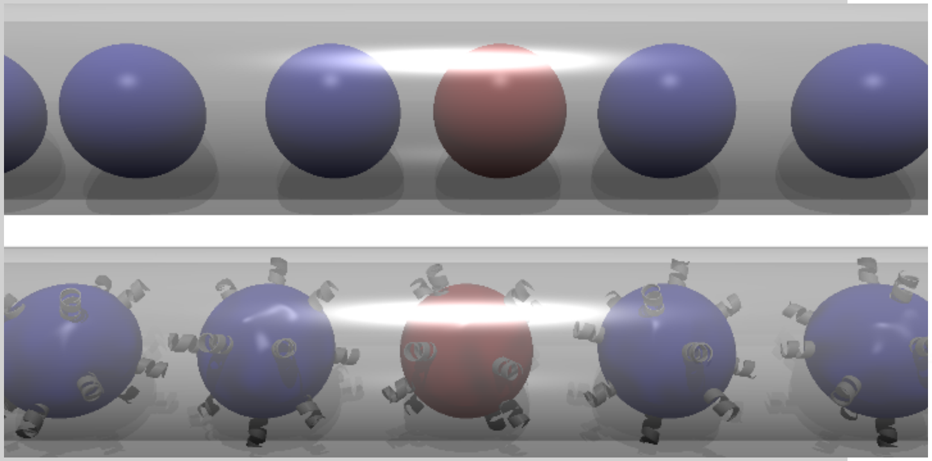
Fractional Langevin equation motion:

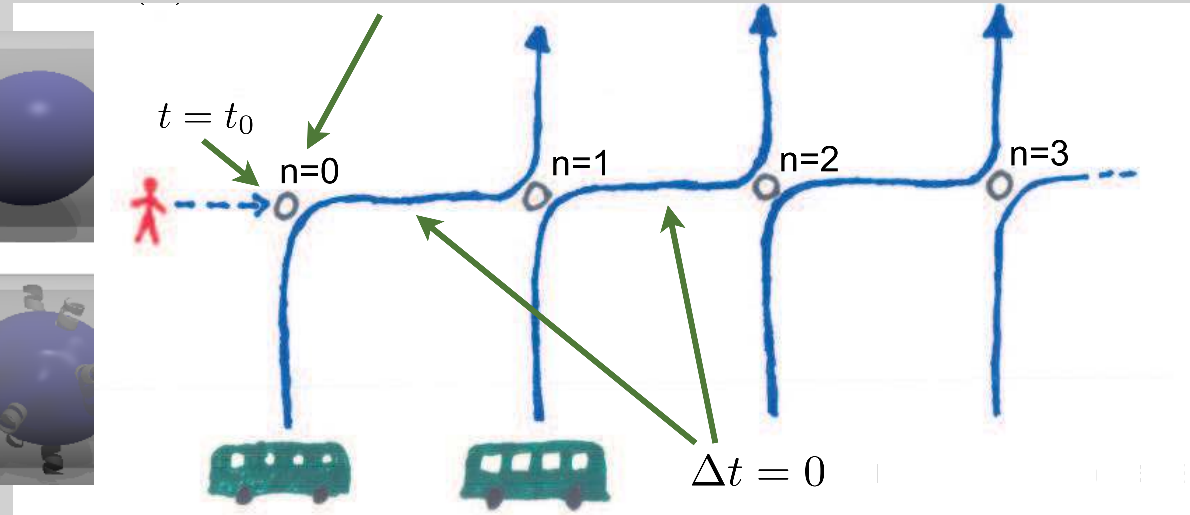
$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = f_{\rm st}(\Delta) + f_{\rm age}(\Delta; t_a, T) \quad \therefore \quad f_{\rm age} \simeq \left[v_0^2 - \langle v^2 \rangle_{\rm th} \right] \Delta^2 t_a^{-2\alpha}, t_a \gg T$$

[J Kursawe, JHP Schulz & RM, PRE (2013)]



Ultraslow dynamics in ageing many-particle systems



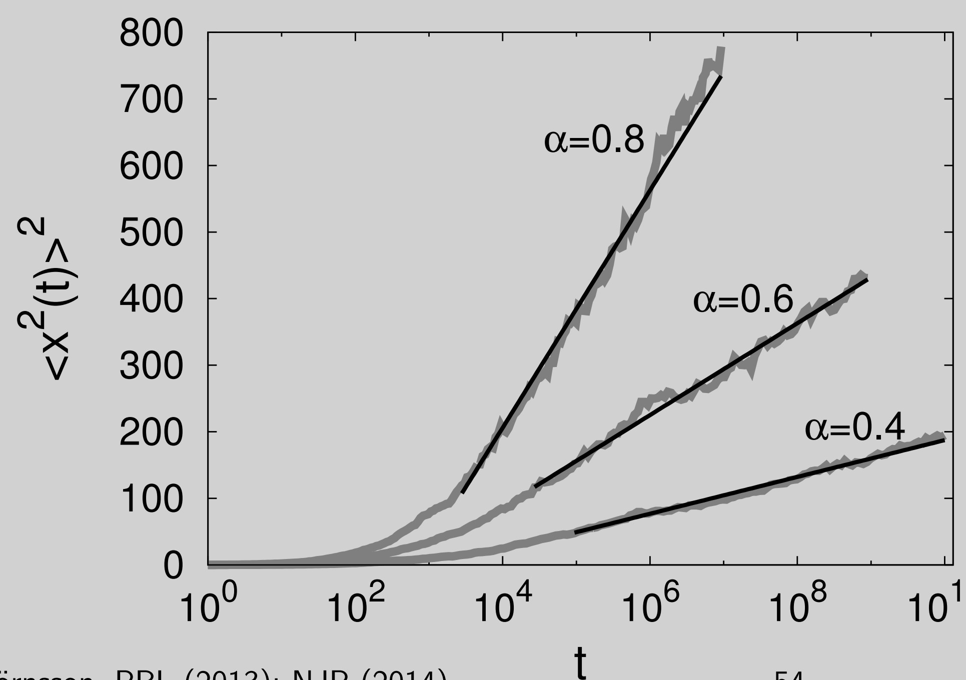


Brownian particles: Harris' law $\langle x^2(t)\rangle \simeq t^{1/2}$

Functionalised particles moving /w $\psi(t) \simeq t^{-1-\alpha}$
$\curvearrowright$ scaling argument: $n \to \log t$:

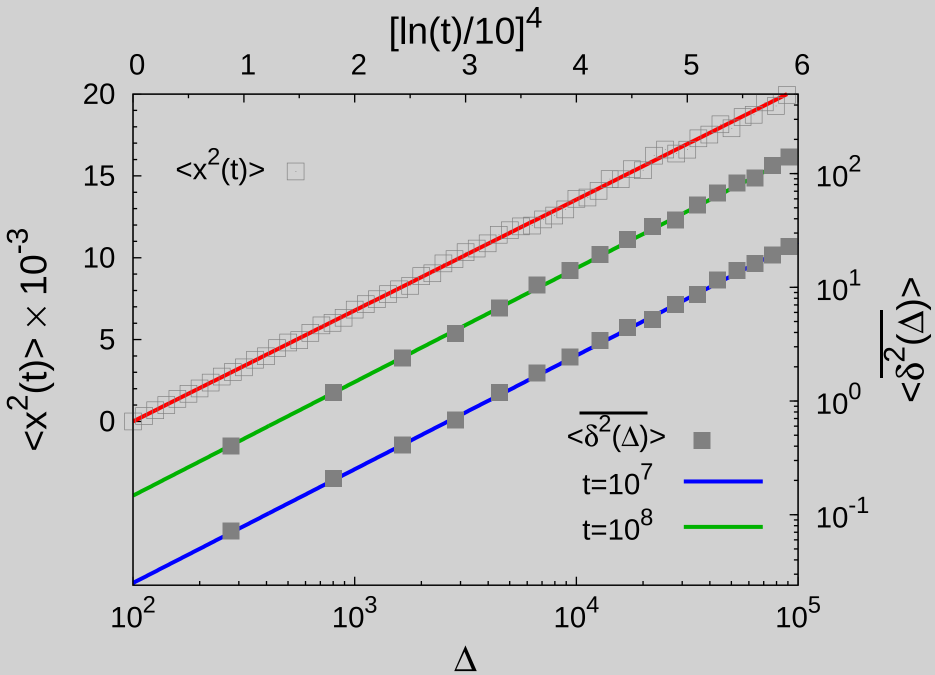
$$\left\langle x^2(t)\right\rangle \simeq \log^{1/2}\left(\frac{t}{t_0}\right)$$

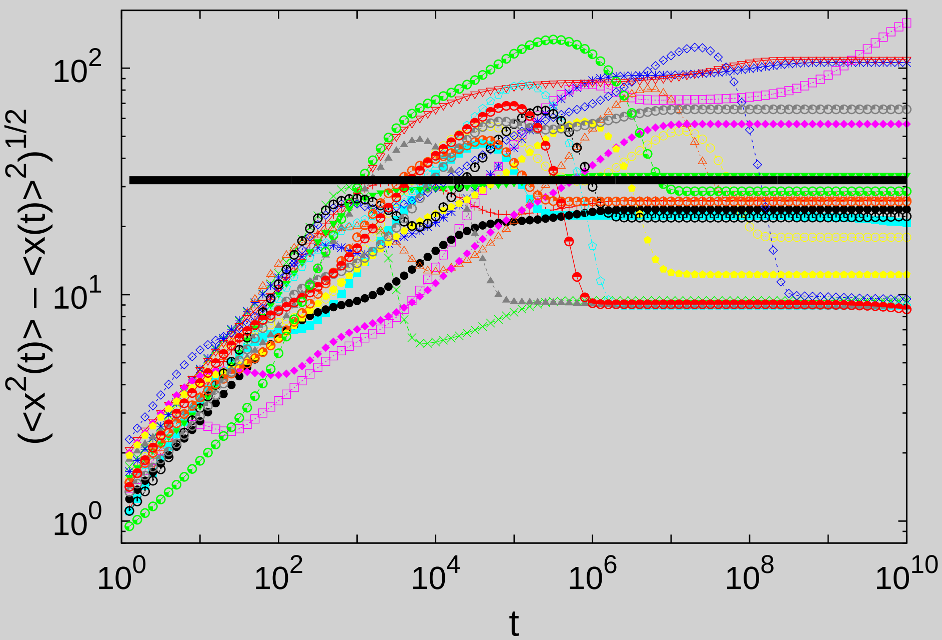
WEB: $\overline{\delta^2(\Delta)} \simeq (\Delta/T)\log^{1/2} T$

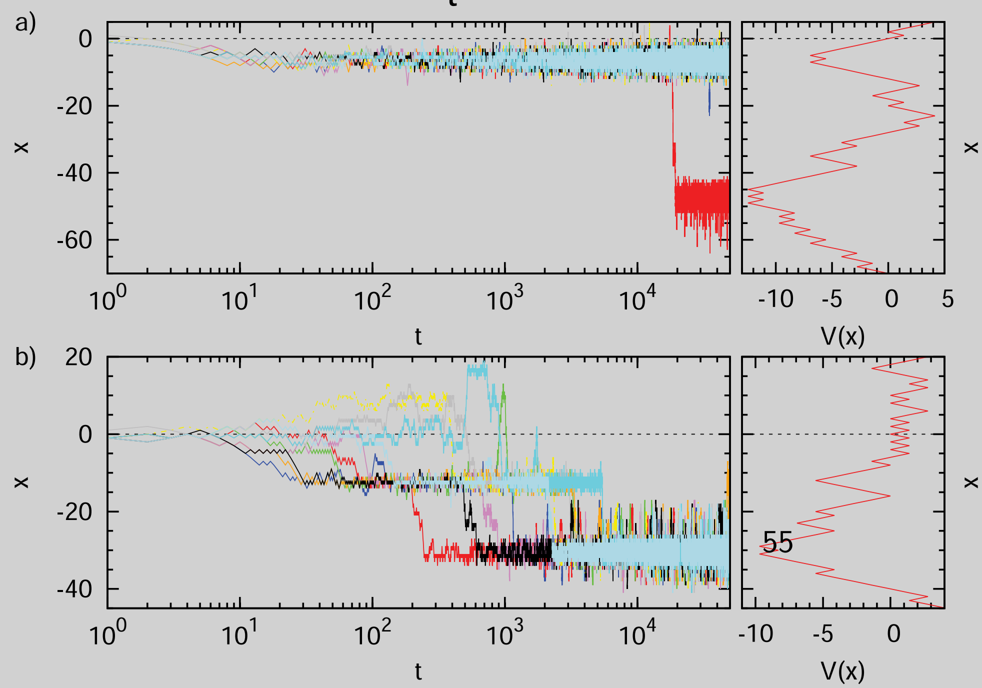


[LP Sanders,] MA Lomholt, L Lizana, [K Fogelmark,] RM & T Ambjörnsson, PRL (2013); NJP (2014)

Logarithmic evolution: Sinai diffusion & mean field CTRW







$$\left\langle \widetilde{\overline{x^2(t)}} \right\rangle \simeq \log^4(t), \quad \left\langle \widetilde{\overline{\delta^2(\Delta)}} \right\rangle \simeq \log^4(T)\frac{\Delta}{T}$$

Mean field approach (comp ultraslow maps):

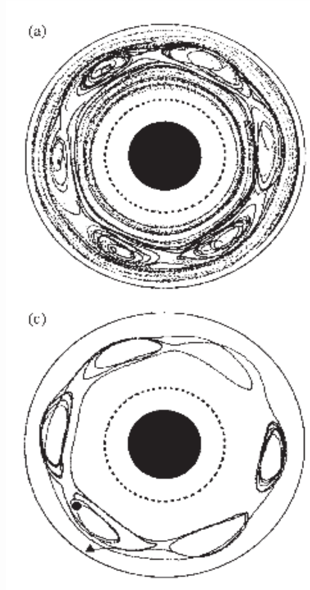
$$\psi(t) \simeq \frac{1}{t \log^4 t}$$

AV Chechkin, H Kantz, A Godec, RM & E Barkai, JPA (2014)

Ultraweak ergodicity breaking: chaos & marine predators

For superdiffusive Lévy walk with $\langle x^2(t)\rangle \simeq t^{3-\alpha}$ and $1<\alpha<2$:

$$\left\langle\overline{\delta^2(\Delta)}\right\rangle_T = \left\langle\overline{\delta x^2}\right\rangle_\infty + T^{3-\alpha}\left[\frac{\alpha-1}{3}\left(\frac{\Delta}{T}\right)^3 - \alpha\left(\frac{\Delta}{T}\right)^2\right]$$

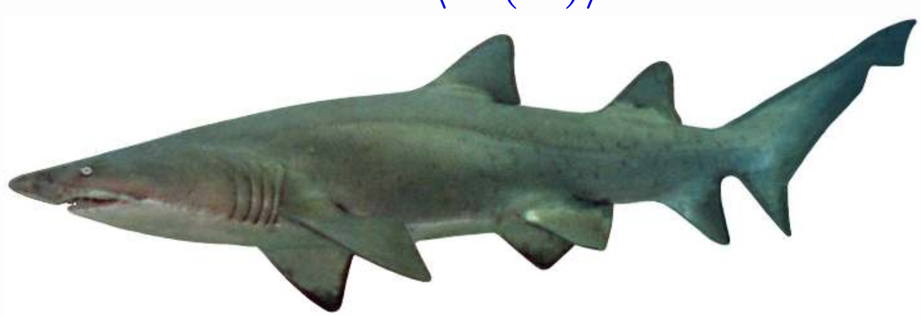


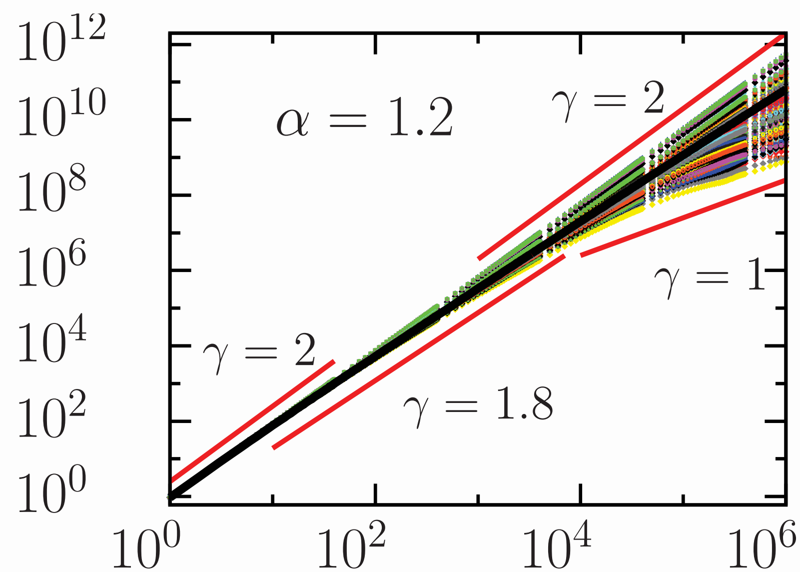
and

$$\left\langle\overline{\delta^2(\Delta)}\right\rangle_\infty = \lim_{T\to\infty}\overline{\delta^2(\Delta)} = 2\left(\frac{(1+\Delta)^{3-\alpha}-1}{(3-\alpha)(2-\alpha)} - \frac{\Delta}{2-\alpha}\right)$$

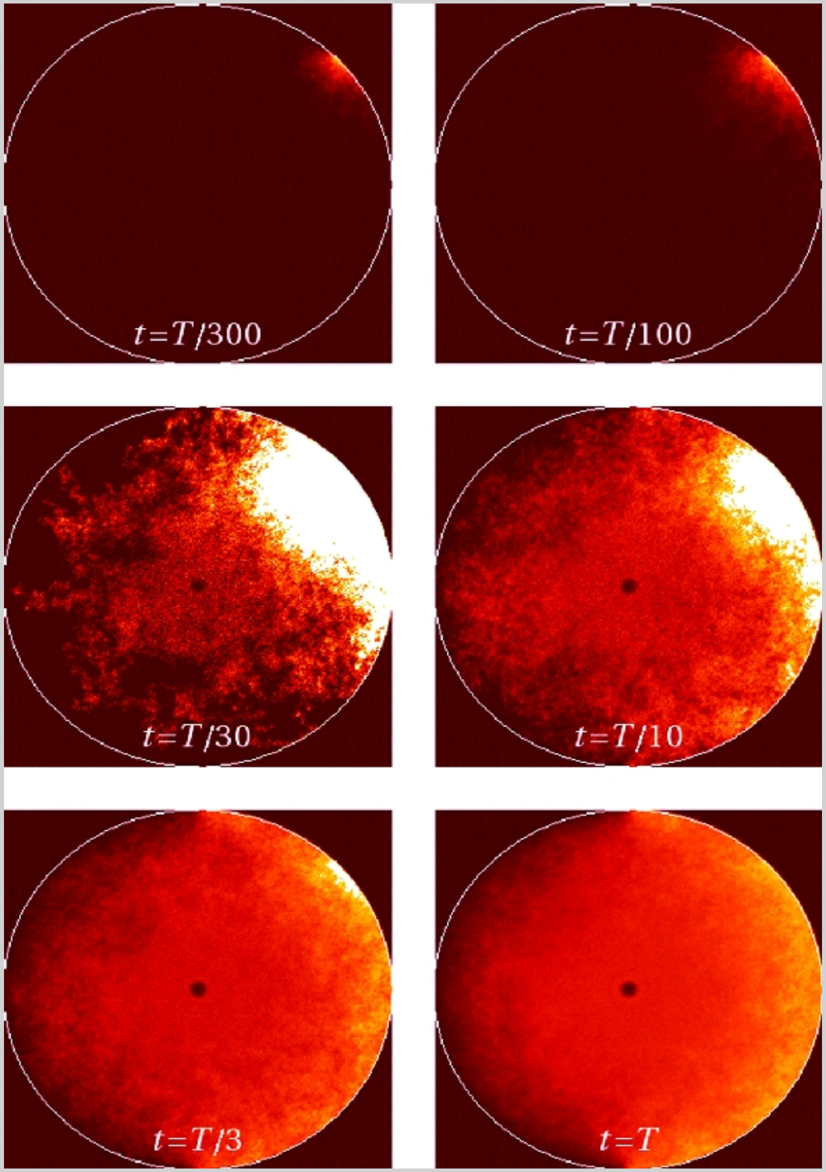
$\mathcal{EB}$ parameter $(T\to\infty)$

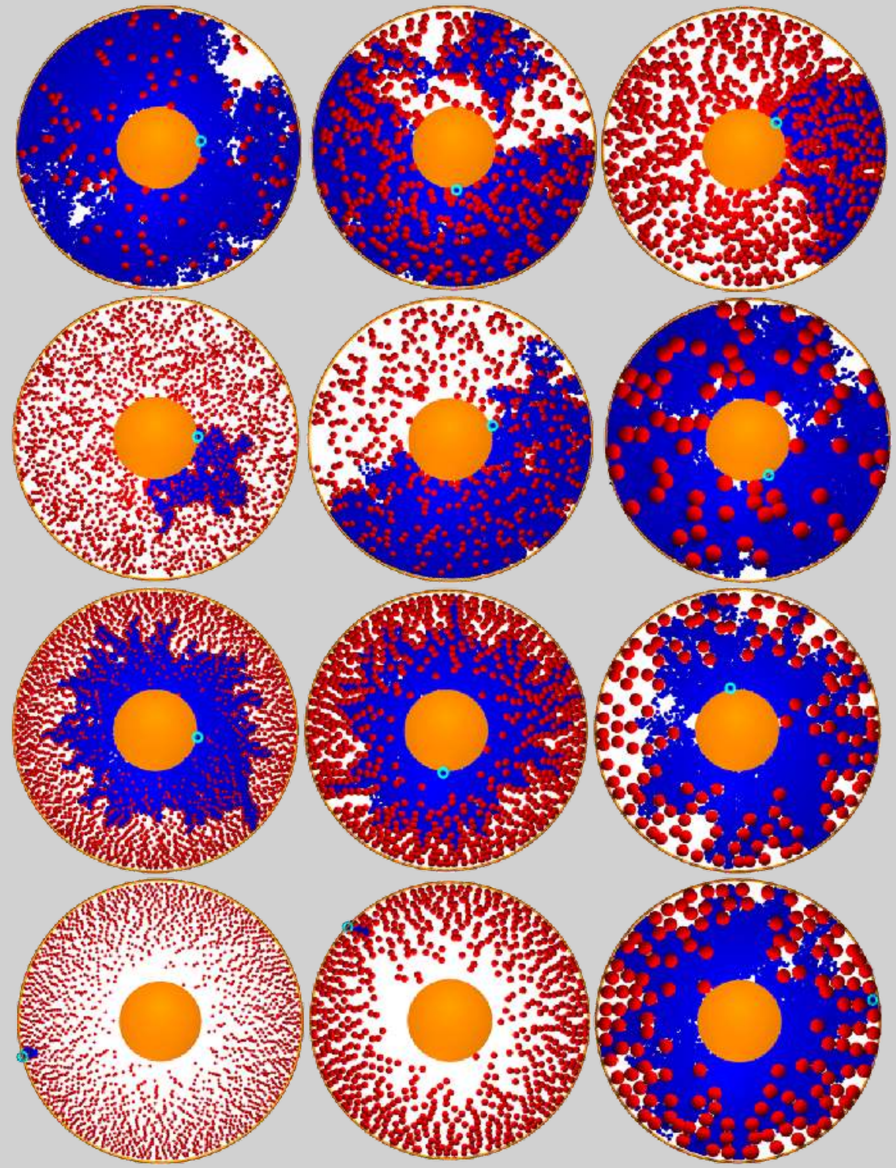
$$\mathcal{EB}(\tau) = \frac{\left\langle\overline{\delta^2(\Delta)}\right\rangle}{\langle x^2(\Delta)\rangle} \sim \frac{1}{\alpha-1}$$





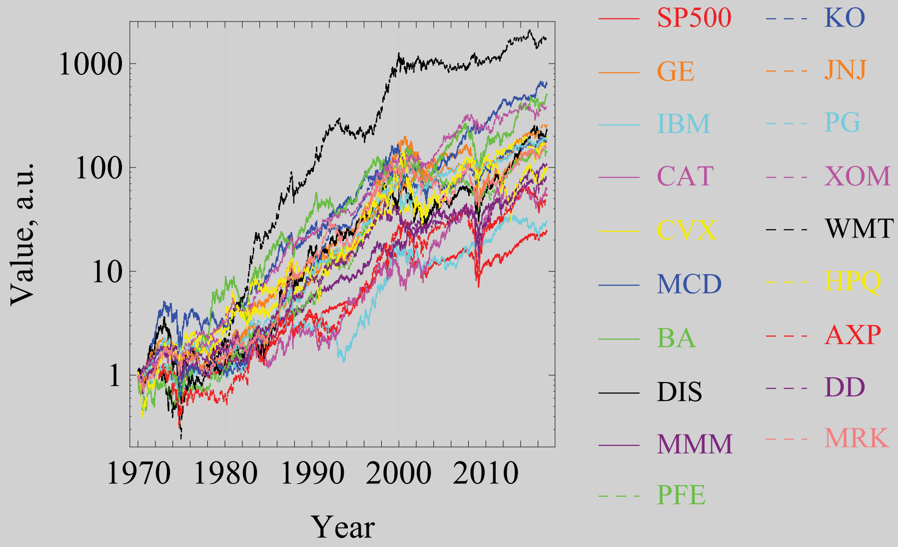
Tracer diffusion in disordered environments

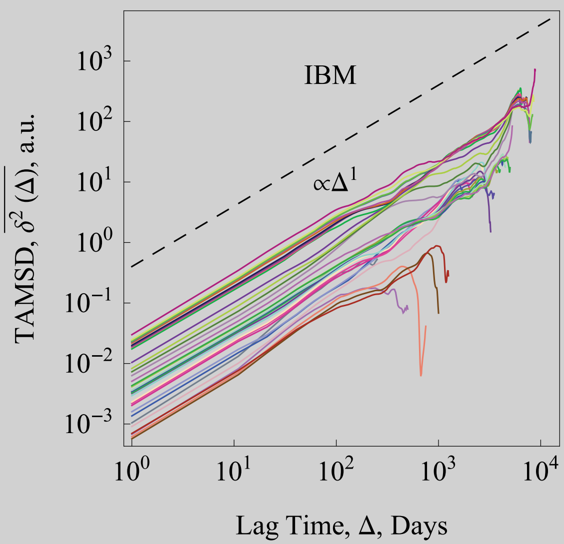


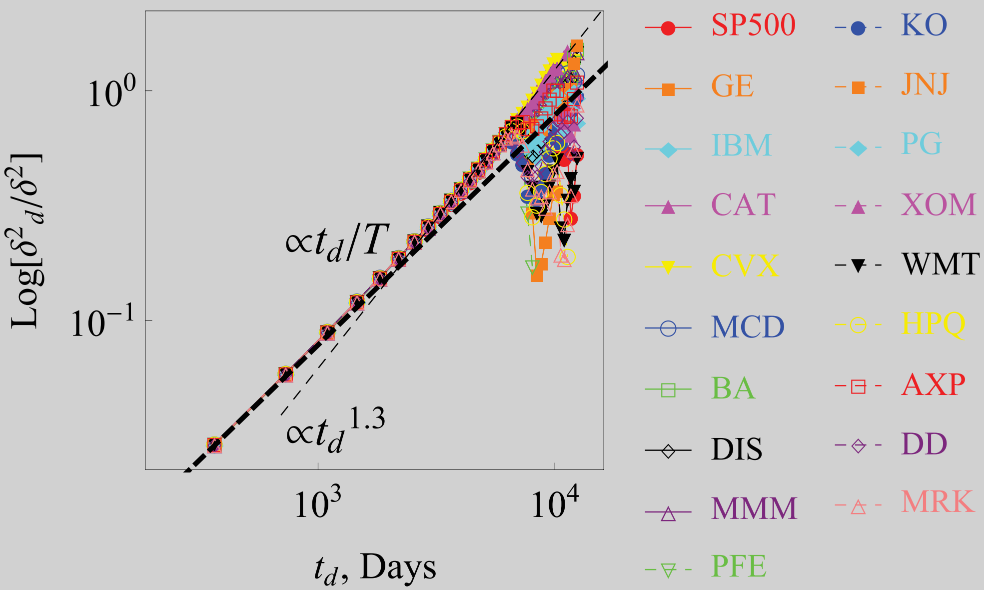


AG Cherstvy, AV Chechkin & RM, Soft Matter (2014); S Ghosh, AG Cherstvy & RM, NJP (2016)

Time averaged & delay time analyses of financial time series







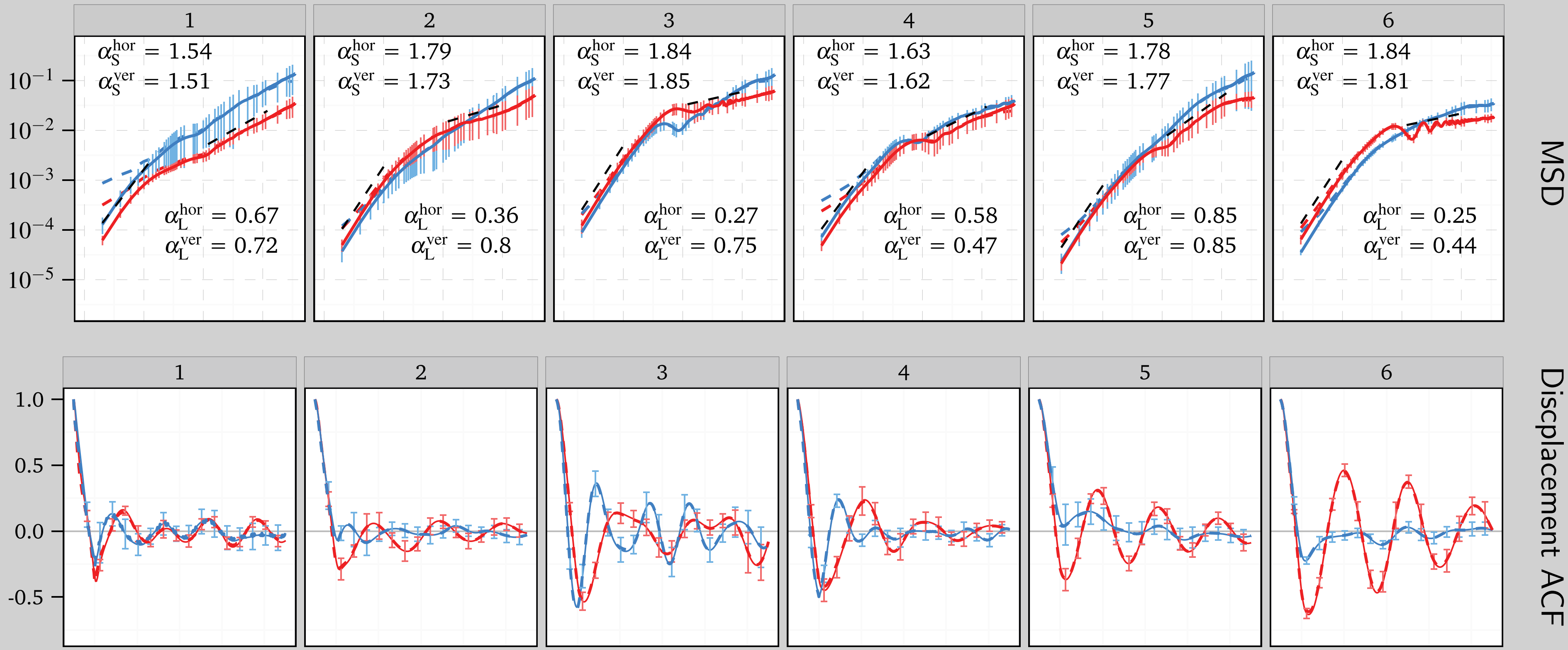
GBM $dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$

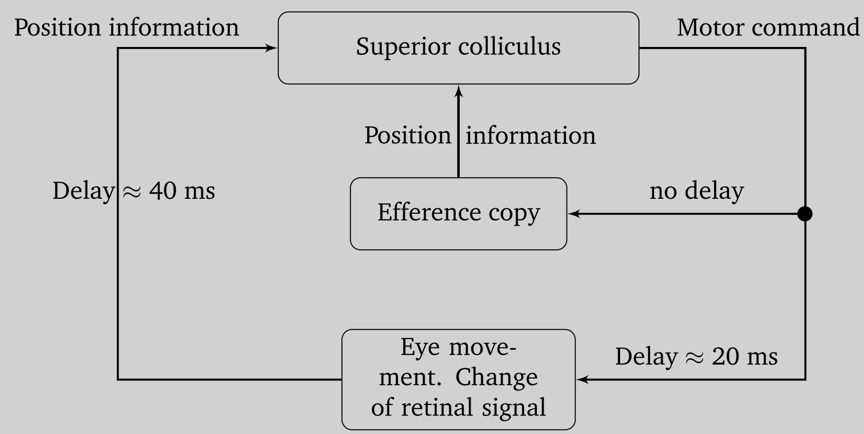
$$\overline{\delta_d^2(\Delta)} = \frac{\int_{t_d}^{T-\Delta} [X(t+\Delta) - X(t)]^2 \, dt}{T - t_d - \Delta} \sim \frac{\Delta}{T - t_d} X_0^2 \left(e^{\sigma^2 T} - e^{\sigma^2 t_d} \right)$$

$$\log \left[\left\langle \overline{\delta_d^2(\Delta, t_d)} \right\rangle \Big/ \left\langle \overline{\delta^2(\Delta)} \right\rangle \right] \sim t_d/T$$

AG Cherstvy, D Vinod, E Aghion, AV Chechkin & RM, NJP (2017)

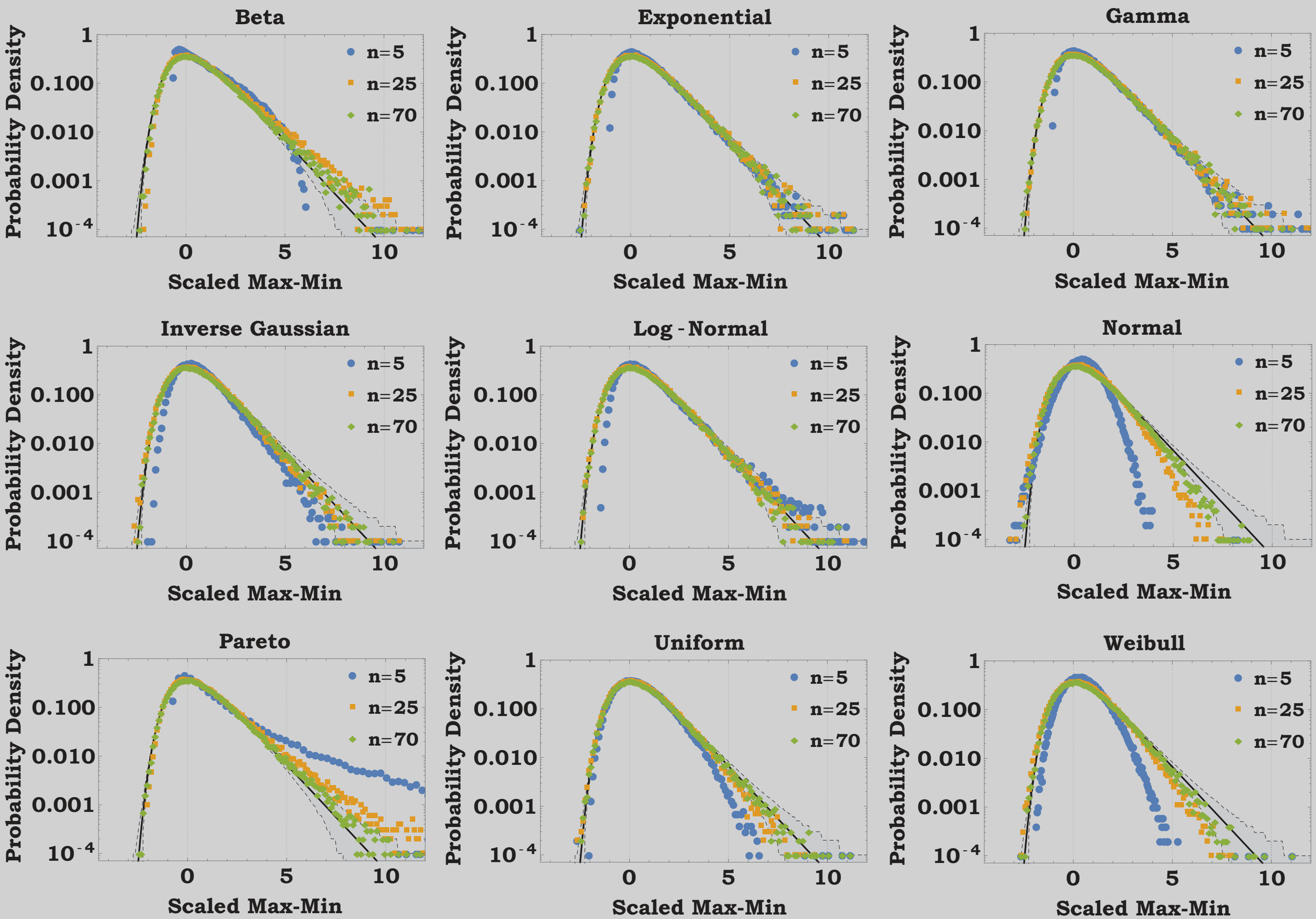
Stochasticity of fixational eye movements





CJJ Herrmann, RM & R Engbert, Sci Rep (2017)

Gumbel universal limit law for Min-Max/Max-Min processes



I Eliazar, RM & S Reuveni, arXiv (2018)

1.34	-0.12
0.13	1.57
-0.91	-0.28
2.13	1.71

Min $\longrightarrow$

-0.12
0.13
-0.91
1.71

Max $\longrightarrow$ 1.71

Manifestations of WEB [RM, JH Jeon, AG Cherstvy & E Barkai, PCCP (2014)]

	$\langle x^2(t)\rangle$	$\left\langle\overline{\delta^2(\Delta)}\right\rangle$	
Correlated jump lenghts	$\simeq t^3$	$\simeq \Delta^2 T$	V Tejedor & RM, JPA (2010)
$K(x)=K_0\lvert x\rvert^\alpha$	$\simeq t^{2/(2-\alpha)}$	$\simeq \Delta T^{2/(2-\alpha)-1}$	A Cherstvy et al, NJP (2013)
$K(t)\simeq K_0t^{\alpha-1}$	$\simeq t^\alpha$	$\simeq \Delta T^{\alpha-1}$	JH Jeon et al, PCCP (2014)
Subdiffusive CTRW	$\simeq t^\alpha$	$\simeq \Delta T^{\alpha-1}$	Y He et al, PRL (2008)
Correlated waiting times	$\simeq t^{\alpha/(1+\alpha)}$	$\simeq \Delta T^{\alpha/(1+\alpha)-1}$	V Tejedor & RM, JPA (2010)
Ultraslow CTRW	$\simeq \log^\alpha(t)$	$\simeq (\Delta/T)\log^\alpha T$	A Godec et al, JPA (2014)
Sinai (quenched)	$\simeq \log^4(t)$	$\simeq (\Delta/T)\log^4 T$	
Ageing CTRW	$\simeq \log(t)$	$\simeq \Delta\log(T)$	MA Lomholt et al, PRL (2013)
$K(x)=K_0\exp(-2x)$	$\simeq \log^2(t)$	$\simeq (\Delta/T)^{1/2}$	A Cherstvy & RM, PCCP (2013)
Lévy walk	$\simeq A(\alpha)t^{3-\alpha}$	$\simeq \frac{A(\alpha)}{\alpha-1}\Delta^{3-\alpha}$	D Froemberg & E Barkai, PRE (2013) A Godec & RM, PRL/PRE (2013)

Overview articles

I Single particle manipulation & tracking:
C Nørregaard, RM, CM Ritter, K Berg-Sørensen & LB Oddershede, Chem Rev **117**, 4342 (2017)

II Anomalous diffusion models, WEB & ageing:
RM, JH Jeon, AG Cherstvy & E Barkai, Phys Chem Chem Phys **16**, 24128 (2014)

III Ageing renewal theory:
JHP Schulz, E Barkai & RM, Phys Rev X **4**, 011028 (2014)

IIII Anomalous diffusion in membranes:
RM, JH Jeon & AG Cherstvy, Biochimica et Biophysica Acta - Biomembranes **1858**, 2451 (2016)

~~IIII~~ Polymer translocation across (membrane) nanopores:
V Palyulin, T Ala-Nissila & RM, Soft Matter **10**, 9016 (2014)